Math Diversion Problem 172

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Don't ever take a fence down until you know the reason it was put up.
— Chesterton

The YouTube video is found at:

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Source: https://www.youtube.com/watch?v=07Elc3M7Tk4
Title: A Proportional Problem of Ratios | Problem 294
 Presenter: aplusbi
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1 The Problem

Given the relation

$$\frac{b+ai}{a+bi} = \frac{4+i}{1+4i}\,, \tag{1}$$

find the values of a, b over the real numbers.

2 The Solution

One could merely cross multiply to get a relation between a and b. When I did that, I got $\{a+bi | b = 4a, a \in \mathbb{R}\backslash\{0\}\}$. Obviously, we can't have both a and b equal to zero.

Now I want to try a more interesting method. Let

$$z \equiv a + bi\,, \quad \text{and} \quad z_0 \equiv 1 + 4i\,. \tag{2}$$

So, $b + ai$ can be morphed into a function of z:

$$z = a + ib\,, \tag{3a}$$
$$iz = ia - b\,, \tag{3b}$$
$$\overline{iz} = -ia - b\,, \tag{3c}$$
$$-\overline{iz} = b + ia\,, \tag{3d}$$
$$i\bar{z} = b + ia\,. \tag{3e}$$

Hence,

$$\frac{b+ai}{a+bi} = \frac{i\bar{z}}{z}\,. \tag{4}$$

And, if we do something similar to $(4+i)/(1+4i)$, we get

$$\frac{4+i}{1+4i} = \frac{i\bar{z}_0}{z_0}\,. \tag{5}$$

Putting them together, we have that

$$\frac{i\bar{z}}{z} = \frac{i\bar{z}_0}{z_0}\,. \tag{6}$$

On cross multiplying, we get

$$\bar{z}z_0 = z\bar{z}_0 = \overline{\bar{z}z_0}\,. \tag{7}$$

What we have here is a complex number that is equal to its own complex conjugate, implying that the complex number must be pure real, and that means that its imaginary part is zero:

$$\mathrm{Im}\,(\bar{z}z_0) = \mathrm{Im}\,[(a-bi)(1+4i)] = 0\,, \tag{8}$$

which means that

$$4a - b = 0\,. \tag{9}$$

And this is what we got before.