Math Diversion 18

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Abstract

Here we use the unipodal algebra to assist in solving the problem, which is given to us on YouTube. Although I'm referring to the series under the name 'olympiad', the problems are from diverse sources as olympiads, entrance exams, SATs, and the like.

The YouTube video is found at:

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https://www.youtube.com/watch?v=JTt0wjqG7vM
Titled: Harvard University Admission Question ||
    Algebra Exam || 99\% Failed Entrance Test
Presenter: Super Academy
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1 The Problem

Given the relations

$$x^5 + y^5 = 152\,, \tag{1a}$$

$$x + y = 2\,, \tag{1b}$$

find all solutions for x, y over the complex numbers.

Note: The Presenter is good at this kind of solution. If I tried to do it the standard way, I might give us after a couple hours with no solutions. However, the unipodal solution given here is both interesting and relatively straightforward.

2 The Prerequisites: The unipodal algebra

This algebra is formed as the extension of the complex numbers by the number u, where $u^2 = 1$, and u commutes with the complex numbers. The number u is said to be 'unipotent'. The set of numbers constructed this way are the unipodal numbers, a particular such number is called a unipode. The main conjugation operator on unipode a is the unegation operator, written a^-. It does not affect complex numbers, but it sends every u to its negative. Hence, if $a = x + yu$, where x, y are complex numbers, then $a^- = x - yu$. Unegation distributes over addition and multiplication.

The following are some properties that will come in handy:

$$u^2 = 1\,, \tag{2a}$$
$$u_\pm \equiv \tfrac{1}{2}(1 \pm u)\,, \tag{2b}$$
$$u_\pm^2 = u_\pm\,, \tag{2c}$$
$$u = u_+ - u_-\,, \tag{2d}$$
$$u_+ u_- = 0\,, \tag{2e}$$
$$u_+ + u_- = 1\,, \tag{2f}$$
$$u u_+ = u_+\,, \tag{2g}$$
$$u u_- = -u_-\,, \tag{2h}$$
$$(u_\pm)^- = u_\mp\,. \tag{2i}$$

You should prove (2c) – (2i). By the way, these two special unipodes $u_\pm$ square to themselves. Such numbers in a ring are referred to as *idempotents*. In the unipodal numbers they have no inverses. The fact that the unipodal number system is not a field is of little concern to me. In fact, most unipodes have inverses, so long as they are not multiples of one of the idempotents. If one needs field elements, the scalars of the unipodal numbers comprise the field of complex numbers.

Two often-used results are, for complex numbers w, z (which are used to convert unipodes between the bases $\{1, u\}$ and $\{u_+, u_-\}$):

$$w + zu = (w + z)u_+ + (w - z)u_-\,, \tag{3a}$$
$$wu_+ + zu_- = \tfrac{1}{2}(w + z) + \tfrac{1}{2}(w - z)u\,. \tag{3b}$$

Much of the algebraic power of the unipodal algebra comes from 1) it being able to switch the presentation of a unipode between the standard basis and the idempotent basis, the latter basis being well suited for taking powers and roots.

3 The Solution

The first question I asked myself in trying to solve this problem is how I should get some equation or equations of fifth-order that won't be a huge problem to deal with. The unipodal algebra provides us a lot of help with that issue.

Clearly, most obvious thing to do to get an expression for x and y both to the fifth powers is to define a unipode in idempotent form, which can then be easily raised to the fifth power. Now, however, what's obvious and what's usual is not always the best way to proceed, but in this case, it is.

Therefore, let our 'first unipode' be defined as follows:[1]

$$a \equiv xu_+ + yu_- \,. \tag{4}$$

Then, to have x and y each raised to the nth power, we get simply

$$a = x^n u_+ + y^n u_- \,. \tag{5}$$

Anyway, we will need the sum of x and y because we will need soon to use the constraint (1b). To that end, we will simply flip the basis of a in (4) to get

$$a = \tfrac{1}{2}(x+y) + \tfrac{1}{2}(x-y)u \,. \tag{6a}$$
$$= 1 + ku \,. \tag{6b}$$
$$= (1+k)u_+ + (1-k)u_- \,. \tag{6c}$$

where we used constraint (1b), and where

$$k \equiv \tfrac{1}{2}(x-y) \,, \tag{6d}$$

which is introduced simply for ease of handling in future complicated expressions.

So, stop and think about this: Eqs. (1b) and (6d) are a coupled pair of first-order equations, so that, if we could solve for k, we could then easily solve them for x and y.

But before we can do that, we must use the constraint equation (1a). But how? The most obvious thing to do would be to raise both equations (4) and (6c) to the fifth power and hope for the best. And this might work. But a slightly less complicated trick will be used (we'll refer to this claim as 'the setup') is to raise both of them to the fourth power and equate corresponding coefficients, to get

$$x^4 = (1+k)^4 = 1 + 4k + 6k^2 + 4k^3 + k^4 \,, \tag{7a}$$
$$y^4 = (1-k)^4 = 1 - 4k + 6k^2 - 4k^3 + k^4 \,. \tag{7b}$$

Note: It may be interesting to realize that at this point we are already **finished** using the unipodal algebra for this problem! From here on, it's just conventional algebra.

Okay, now here's the payoff of the 'trick' I mentioned earlier. We multiply through by x on (7a) and by y on (7b), yielding

$$x^5 = x + 4xk + 6xk^2 + 4xk^3 + xk^4 \,, \tag{8a}$$
$$y^5 = y - 4yk + 6yk^2 - 4yk^3 + yk^4 \,. \tag{8b}$$

[1] I define the term 'first unipode' as the point of entrance of the real or complex numbers into the unipodal algebra. There are often a variety of choices for this first unipode that could work, also usually one that stands out as superior. The fun is, finding it!

There's nothing left to do but to add these equations together and hope we get a solvable equation in k:

$$x^5 + y^5 = (x+y) + 4(x-y)k + 6(x+y)k^2 + 4(x-y)k^3 + (x+y)k^4\,,$$
$$= 2 + 8k^2 + 12k^2 + 8k^4 + 2k^4\,. \tag{9a}$$

On using (1a) and simplifying, we get that

$$k^4 + 2k^2 - 15 = 0\,, \tag{10}$$

and with that we note that we have finished using all the given information. There's nothing left to do now but try to use the equation to find the solutions.

Using the quadratic equation on solving for the unknown k^2, gives

$$k^2 = \frac{-2 \pm \sqrt{64}}{2} = -1 \pm 4\,. \tag{11}$$

We'll begin with the + root: $k^2 = 3$. We have the simultaneous set:

$$k = +\sqrt{3} = \tfrac{1}{2}(x-y)\,, \tag{12a}$$
$$2 = x + y\,. \tag{12b}$$

Solving these, we get

$$x = 1 + \sqrt{3}\,, \quad y = 1 - \sqrt{3}\,, \tag{13a}$$
$$x = 1 - \sqrt{3}\,, \quad y = 1 + \sqrt{3}\,. \tag{13b}$$

Next, we go with the $-$ root to (11): $k^2 = -5$. We have the couple:

$$k = \pm i\sqrt{5} = \tfrac{1}{2}(x-y)\,, \tag{14a}$$
$$2 = x + y\,. \tag{14b}$$

Solving these, we get

$$x = 1 + i\sqrt{5}\,, \quad y = 1 - i\sqrt{5}\,, \tag{15a}$$
$$x = 1 - i\sqrt{5}\,, \quad y = 1 + i\sqrt{5}\,. \tag{15b}$$

4 Conclusion

I can explain why we have ended up with only four solutions to a fifth degree equation. A fourth-order equation in x is obtained by eliminating y between the two given equations (1a) and (1b), the terms x^5 and $-x^5$ canceling out.

As a postscript, the fourth-order equation we get by adding together (7a) and (7b), yielding

$$x^4 + y^4 = 2 + 12k^2 + 2k^4\,, \tag{16}$$

turns out to be a mere identity in the variable x.