Math Diversion Problem 187

P. Reany

November 25, 2024

With me, everything turns into mathematics.
— Rene Descartes
(P.S. I calculate; therefore I am.)

The YouTube video is found at:

```
Source: https://www.youtube.com/watch?v=FXEFFSDwglE
Title: A Very Exponential Power Tower
         | Problem 272
 Presenter: aplusbi
```

1 The Problem

Given the relation

$$z^{z^{z+1}} = 4\,, \tag{1}$$

find the values of z. (Skip down to the solution, if you like.)

2 Basics of Complex Numbers

Typically, we find a generic complex number denoted by the letter z, but one is free to choose other letters, as well. So, if z is a complex number, in general it has both real and imaginary parts:

$$z = a + bi\,, \tag{2}$$

where a, b are real components of basis vectors $1, i$. But they are also expressed as, respectively, the 'real' and 'imaginary' components of z.

Complex conjugation of complex number z is an operation that leaves real numbers alone but replaces the unit imaginary i with its negative, i.e., $-i$. The symbols most often used to represent complex conjugation are the $*$ and the overbar. I'll usually use the overbar. Thus, the complex conjugate of z in (2) is

$$\overline{z} = a - bi\,. \tag{3}$$

Obviously, the complex conjugation of a pure real number has no effect.

A funny thing happens when we multiply a complex number by its conjugate:

$$z\overline{z} = (a + bi)(a - bi) = a^2 + b^2 \,. \tag{4}$$

So, $z\overline{z}$ is zero if and only if $z = 0$, otherwise, it's a positive real number.

Another funny thing happens when we add a complex number and its conjugate: we also get a real number. Let's see.

$$z + \overline{z} = (a + bi) + (a - bi) = 2a \,. \tag{5}$$

Why do we care about this? Because sometimes we need to map complex numbers into the real numbers to get information on the complex numbers. This problem will show you that.

I'm not going to prove this here, but every complex number can be expressed in exponential (or polar) form:

$$z = a + bi = \sqrt{a^2 + b^2} e^{i\theta} = (z\overline{z})^{1/2} e^{i\theta} = r\, e^{i\theta} \,, \tag{6}$$

where we can think of r as the length of the complex numbers z or $\overline{z}$.

$$r \equiv (z\overline{z})^{1/2} \quad \text{or} \quad r^2 = z\overline{z} = |\, z \,|^2 \,. \tag{7}$$

So, it will be good to know all this stuff in this section before you attempt to follow my solutions to these complex variables problems.

By the way, the complex numbers are what's called a *field*, so they can be added, subtracted, multiplied, and divided by each other (except you can't divide by zero, as usual). And, therefore, you can apply the quadratic formula to them! (Yay!)

Lemma 1: If a complex number z is equal to its own conjugate $z = \overline{z}$, it's real.

Lemma 2: If a complex number z is complex conjugated twice then there's no change: $\overline{\overline{z}} = z$.

Lemma 3: The complex conjugated of a product or a sum is the product or sum of the complex conjugates: $\overline{z_1 z_2} = \overline{z}_1 \overline{z}_2$ and $\overline{z_1 + z_2} = \overline{z}_1 + \overline{z}_2$.

Lemma 4: If $s, t \in \mathbb{R}$ and $z = s + ti$ then

$$i\overline{z} = t + si \,. \tag{8}$$

3 Basics of Complex Numbers with Trig Functions

Let's begin with the Euler relations:

$$\cos\theta + i\sin\theta = e^{i\theta} \,, \tag{9a}$$

$$\cos\theta - i\sin\theta = e^{i\theta} \,, \tag{9b}$$

Next, let's invert them:

$$\cos\theta = \frac{1}{2}(e^{i\theta} + e^{i\theta})\,, \tag{10a}$$

$$\sin\theta = \frac{1}{2i}(e^{i\theta} - e^{i\theta})\,, \tag{10b}$$

where, in the above cases, I used the usually understood real variable θ, but that can be replaced by the complex variable z. In fact, soon we will do so.

Okay, how to represent $\tan z$ by exponentials?

$$\tan z = \frac{\sin z}{\cos z} = \frac{1}{i}\frac{e^{iz} - e^{-iz}}{e^{iz} + e^{-iz}}\,. \tag{11}$$

4 The Solution

We can rewrite (1) to

$$z^{z^{z+1}} = 2^2\,, \tag{12}$$

revealing the base to be 2. Therefore, I'll introduce the change of variable

$$z = 2^\alpha\,. \tag{13}$$

So, (12) becomes

$$(2^\alpha)^{(2^\alpha)^{(2^\alpha)+1}} = 2^2\,, \tag{14}$$

which simplifies down to

$$(2^\alpha)^{2^{\alpha[(2^\alpha)+1]}} = 2^2\,, \tag{15}$$

and once again

$$2^{\alpha(2^{\alpha[2^\alpha+1]})} = 2^2\,. \tag{16}$$

On equating exponents, we get

$$\alpha 2^{\alpha[2^\alpha+1]} = 2\,. \tag{17}$$

This can be rewritten as

$$\left(\alpha 2^\alpha\right) 2^{\alpha 2^\alpha} = 2\,. \tag{18}$$

It's time to start thinking about the Lambert W function! Let

$$h = \alpha 2^\alpha\,. \tag{19}$$

Then (18) becomes

$$h 2^h = 2\,. \tag{20}$$

Let's introduce the variable y, such that

$$e^y = 2^h\,, \tag{21}$$

then

$$y = h \ln 2 \quad \text{and} \quad h = y / \ln 2 \,, \tag{22}$$

Then (20) becomes

$$ye^y = 2 \ln 2 \,. \tag{23}$$

Thus,

$$y = W(2 \ln 2) \,. \tag{24}$$

Therefore,

$$h_0 = \frac{W(2 \ln 2)}{\ln 2} \,. \tag{25}$$

where the subscript 0 indicates that h is now a known quantity. This takes us back to (19), which we'll now write as

$$\alpha 2^\alpha = h_0 \,, \tag{26}$$

and once again we invoke the Lambert W function. Set

$$e^\beta = 2^\alpha \,, \tag{27}$$

then

$$\beta = \alpha \ln 2 \quad \text{and} \quad \alpha = \beta / \ln 2 \,. \tag{28}$$

Then (26) becomes

$$\beta e^\beta = h_0 \ln 2 \,. \tag{29}$$

Therefore,

$$\beta = W(h_0 \ln 2) \,. \tag{30}$$

From (28) we have that

$$\alpha = \beta / \ln 2 = \frac{W(h_0 \ln 2)}{\ln 2} \,. \tag{31}$$

And this takes us back to (13)

$$z = 2^{\frac{W(W(2 \ln 2))}{\ln 2}} \,. \tag{32}$$

Wikipedia claims that $W_0(2 \ln 2) = \ln 2$. So,

$$z = 2^{\frac{W(\ln 2)}{\ln 2}} \,. \tag{33}$$

Or we can say that

$$z = e^\beta = e^{W(W(2 \ln 2))} = e^{W(\ln 2)} \,. \tag{34}$$