Math Diversion Problem 196

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Small moves, Ellie. Small moves.
— from the movie, *Contact*

The YouTube video is found at:

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Source: https://www.youtube.com/watch?v=rqDnyMZ9oiU
Title: Can You Solve? | America's Hard Maths Word Problem
 Presenter: Daily Mathematics
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1 The Problem

Word Problem:
A man invested a sum of \$280, partly at 5% and partly at 4%. If the total interest is \$12.90 per annum, find the amount invested at 5%.

If you're interested, see my many articles on solving algebra word problems at:

https://www.advancedmath.org/Math/AlgebraWordProblems.html

(Skip down to the solution, if you like.)

2 Basics of Complex Numbers

Typically, we find a generic complex number denoted by the letter z, but one is free to choose other letters, as well. So, if z is a complex number, in general it has both real and imaginary parts:

$$z = a + bi\,, \tag{1}$$

where a, b are real components of basis vectors $1, i$. But they are also expressed as, respectively, the 'real' and 'imaginary' components of z.

Complex conjugation of complex number z is an operation that leaves real numbers alone but replaces the unit imaginary i with its negative, i.e., $-i$. The symbols most often used to represent complex conjugation are the $*$ and the overbar. I'll usually use the overbar. Thus, the complex conjugate of z in (1) is

$$\overline{z} = a - bi\,. \tag{2}$$

Obviously, the complex conjugation of a pure real number has no effect.

A funny thing happens when we multiply a complex number by its conjugate:

$$z\overline{z} = (a + bi)(a - bi) = a^2 + b^2\,. \tag{3}$$

So, $z\overline{z}$ is zero if and only if $z = 0$, otherwise, it's a positive real number.

Another funny thing happens when we add a complex number and its conjugate: we also get a real number. Let's see.

$$z + \overline{z} = (a + bi) + (a - bi) = 2a\,. \tag{4}$$

Why do we care about this? Because sometimes we need to map complex numbers into the real numbers to get information on the complex numbers. This problem will show you that.

I'm not going to prove this here, but every complex number can be expressed in exponential (or polar) form:

$$z = a + bi = \sqrt{a^2 + b^2}e^{i\theta} = (z\overline{z})^{1/2}e^{i\theta} = r\,e^{i\theta}\,, \tag{5}$$

where we can think of r as the length of the complex numbers z or $\overline{z}$.

$$r \equiv (z\overline{z})^{1/2} \quad \text{or} \quad r^2 = z\overline{z} = |z|^2\,. \tag{6}$$

So, it will be good to know all this stuff in this section before you attempt to follow my solutions to these complex variables problems.

By the way, the complex numbers are what's called a *field*, so they can be added, subtracted, multiplied, and divided by each other (except you can't divide by zero, as usual). And, therefore, you can apply the quadratic formula to them! (Yay!)

Lemma 1: If a complex number z is equal to its own conjugate $z = \overline{z}$, it's real.

Lemma 2: If a complex number z is complex conjugated twice then there's no change: $\overline{\overline{z}} = z$.

Lemma 3: The complex conjugated of a product or a sum is the product or sum of the complex conjugates: $\overline{z_1 z_2} = \overline{z}_1\overline{z}_2$ and $\overline{z_1 + z_2} = \overline{z}_1 + \overline{z}_2$.

Lemma 4: If $s, t \in \mathbb{R}$ and $z = s + ti$ then

$$i\overline{z} = t + si\,. \tag{7}$$

3 Basics of Complex Numbers with Trig Functions

Let's begin with the Euler relations:

$$\cos\theta + i\sin\theta = e^{i\theta}\,, \tag{8a}$$

$$\cos\theta - i\sin\theta = e^{i\theta}\,, \tag{8b}$$

Next, let's invert them:

$$\cos\theta = \frac{1}{2}(e^{i\theta} + e^{i\theta})\,, \tag{9a}$$

$$\sin\theta = \frac{1}{2i}(e^{i\theta} - e^{i\theta})\,, \tag{9b}$$

where, in the above cases, I used the usually understood real variable θ, but that can be replaced by the complex variable z. In fact, soon we will do so.

Okay, how to represent $\tan z$ by exponentials?

$$\tan z = \frac{\sin z}{\cos z} = \frac{1}{i}\frac{e^{iz} - e^{-iz}}{e^{iz} + e^{-iz}}\,. \tag{10}$$

4 Problem-Solving Aids (Heuristics)

One way to get really confused or even lost in solving a word problem is to do too much in a single step. The fix to this is to slow down and be very intentional about everthing you write down. Develop the equations slowly. This philosophy should be apparent in the manner of the development of the heuristics and the problem solving technique employed below.

When I approach a word problem, I have a set of heuristics to guide me through the process:

1. If there are any totals or parts of a total lying around, put them into an equation (or into an inequality, if appropriate). Then, note that the following rule generates an equation: *Every total is equal to the sum of all its parts.*

$$\text{Total} = \sum_i \text{Parts}_i\,. \tag{11}$$

2. Is there some invariant Inv evidently holding from the initial state to the final state of a before-and-after process? If so, write

$$\text{Inv}_{\text{before}} = \text{Inv}_{\text{after}}\,. \tag{12}$$

And that's another equation to work with. For example, say we have a beaker containing 50 ml of salt solution (in water), to which we add 10 ml of water. Now, although the amount of water in the beaker has not remained invariant throughout this procedure, the amount of salt has. So, you can write an equation out of that invariance!

3. Is there a common or problem-specific formula to use? Such as from physics, chemistry, etc., or from mathematics, like from geometry or from number theory, such as for the summation of a series or for a weighted average of a set of numbers, or the greatest common factor or least common multiple of two or more numbers, and so on. For example, the area of a triangle $A = \frac{1}{2}bh$ is an equation. Furthermore, is there a problem-specific relationship given in the problem, such "the base of the triangle is one-third its height." That's an equation!

4. Is there a proportion given? A proportion is the stated equality of two ratios, such as

$$\frac{A}{B} = \frac{C}{D}\,. \tag{13}$$

5. Are there one or more linear or quadratic equations given? If so, write them down.

When we've collected as many equations as we have unknowns, we should be ready to solve the system simultaneously. (However, no one of these equations should be derivable from the others of the system.)

5 The Solution

Restatement of the problem: A man invested a sum of \$280, part at 5% and part at 4%. If the total interest is \$12.90 per annum, find the amount invested at 5%.

So, my first assumption is that these two investment have accrued interest over a single year. That simplifies things. Now, in applying my set of heuristics to this problem, I immediately uncover two distinct totals. The first is the total amount of money invested (TI), and the second is the total amount of interest accrued at the end of the year.

Let's write down an equation for each of these totals, setting them equal to the sum of their parts, as we go. The first total will be the total amount of money invested:

$$TI = (\text{Amount invested at } 5\%) + (\text{Amount invested at } 4\%)\,. \tag{14}$$

If we set I_5 as the amount invested at 5% and I_4 as the amount invested at 4%, then this last equation can be rewritten as

$$280 = I_5 + I_4\,. \tag{15}$$

The second total given is the total amount of interest accrued over the year, that being \$12.90. I will refer to this total as A. The total interest accrued is the sum of that part accrued on the investment at 5% (Q_5) and the part invested at 4% (Q_4). Thus, the second equation takes the form

$$A = Q_5 + Q_4 \,. \tag{16}$$

Now, the amount of interest accrued at a given percentage is equal to the amount of money invested at that interest, times the interest rate, as a decimal amount. Thus, (16) becomes

$$12.90 = 0.05 I_5 + 0.04 I_4 \,, \tag{17}$$

where I have suppressed the units, but only after making sure that all the units are correct.

Now, since we have been asked to determine only I_5, we can eliminate I_4 between (15) and (17), to get

$$12.90 = 0.05 I_5 + 0.04(280 - I_5) \,, \tag{18}$$

where the answer is

$$I_5 = 170 \,, \tag{19}$$

which is in dollars.