# Math Diversions, Problem 2

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## 1 Problem

This problem is found on the YouTube channel **PreMath**, from December 23, 2021. My solution here is a little different from that given by the presenter.

**Statement of the problem:**

Given the relations

$$\frac{x^3+y^3}{x+y} = 7\,, \tag{1}$$

and

$$\frac{x^3-y^3}{x-y} = 19\,, \tag{2}$$

Find the possible real (integer) values of $x$ and $y$.

**Solution to the problem:**

My experience in solving for the roots to the general cubic equation led me try to solve this problem by multiplying the two equations together, as

$$\frac{x^3+y^3}{x+y}\frac{x^3-y^3}{x-y} = 7\times 19\,, \tag{3}$$

which reduces to

$$\frac{x^6-y^6}{x^2-y^2} = 133\,. \tag{4}$$

Now for a simplifying variable substitution:

$$\alpha = x^2\,, \quad \beta = y^2\,, \tag{5}$$

to get

$$\frac{\alpha^3-\beta^3}{\alpha-\beta} = 133\,. \tag{6}$$

But we have a formula for the difference of two cubes, namely

$$\alpha^3-\beta^3 = (\alpha-\beta)(\alpha^2+\alpha\beta+\beta^2)\,. \tag{7}$$

On applying this identity to the equation before it, we get that

$$\alpha^2 + \alpha\beta + \beta^2 = 133\,. \tag{8}$$

Let's put this equation in standard quadratic form and assume that $\alpha$ is our unknown to solve for. Then

$$\alpha^2 + \beta\alpha + (\beta^2 - 133) = 0\,. \tag{9}$$

Then, by use of the quadratic formula:

$$\alpha = \frac{-\beta \pm \sqrt{\beta^2 - 4(\beta^2 - 133)}}{2}\,, \tag{10}$$

which can be rewritten as

$$\alpha = \frac{-\beta \pm \sqrt{532 - 3\beta^2}}{2}\,. \tag{11}$$

Now, I'm going to make a simplifying assumption that both $\alpha$ and $\beta$ are positive integers, and this implies that the discriminant of the squareroot of the last equation is a perfect square, which we should be able to find by exhaustive search on values of $\beta$. I say this with confidence because if the discriminant gets too large then the squareroot goes imaginary, which we have assumed must not happen.

Because of how $\alpha$ and $\beta$ were defined in terms of $x$ and $y$, we need only consider perfect-square values of $\alpha$ and $\beta$:

| $\beta$ | $532 - 3\beta^2$ | $\sqrt{532 - 3\beta^2}$ | $\alpha = (-\beta + \sqrt{532 - 3\beta^2})/2$ |
|---|---|---|---|
| 1 | 529 | 23 | 11 |
| 4 | 484 | 22 | 9 ✓ |
| 9 | 289 | 17 | 4 ✓ |
| 16 | $-236$ | — | — |

So, we have our two perfect squares to work with. We can choose $\alpha = 4$ and $\beta = 9$, which gives us for values of $x$ and $y$

$$x = \pm 2 \quad \text{and} \quad y = \pm 3\,, \tag{12}$$

which gives us the four pairs of solutions:

$$(2, 3)\,,\ (2, -3)\,,\ (-2, 3)\,,\ (-2, -3)\,, \tag{13}$$

but because of the symmetry of how $\alpha$ and $\beta$ entered into (8), we can also switch the order of these ordered pairs to get

$$(3, 2)\,,\ (-3, 2)\,,\ (3, -2)\,,\ (-3, -2)\,. \tag{14}$$

At this point, I entered the two original equations and input all eight of these ordered pairs into WolframAlpha.com, but found that only four of them worked. These four were:

$$(2,3)\,,\ (-2,-3)\,,\ (3,2)\,,\ (-3,-2)\,. \tag{15}$$

Done.