Math Diversion Problem 20

P. Reany

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Abstract

Here we use the unipodal algebra to assist in solving the problem, which is given to us on YouTube. Although I'm referring to the series under the name 'olympiad', the problems are from diverse sources as olympiads, entrance exams, SATs, and the like.

The YouTube video is found at:

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https://www.youtube.com/watch?v=C-0anvb3D4k
 Titled: Japan | A nice Math Olympiad Algebra Problem
              | Find x=? & y=?
 Presenter: Super Academy
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1 The Problem

Given the relation

$$x^2 - y^2 = \sqrt{10}\,, \tag{1}$$

$$xy = \sqrt{10}\,, \tag{2}$$

solve for $x + y$ over the complex numbers. (If you really want to, you can solve for x and y as well.)

2 The Prerequisites: The unipodal algebra

This algebra is formed as the extension of the complex numbers by the number u, where $u^2 = 1$, and u commutes with the complex numbers. The number u is said to be 'unipotent'. The set of numbers constructed this way are the unipodal numbers, a particular such number is called a unipode. The main conjugation operator on unipode a is the unegation operator, written a^-. It does not affect complex numbers, but it sends every u to its negative. Hence, if $a = x + yu$, where x, y are complex numbers, then $a^- = x - yu$. Unegation distributes over addition and multiplication.

The following are some properties that will come in handy:

$$u^2 = 1\,, \tag{3a}$$

$$u_\pm \equiv \tfrac{1}{2}(1 \pm u)\,, \tag{3b}$$

$$u_\pm^2 = u_\pm\,, \tag{3c}$$

$$u = u_+ - u_-\,, \tag{3d}$$

$$u_+ u_- = 0\,, \tag{3e}$$

$$u_+ + u_- = 1\,, \tag{3f}$$

$$u u_+ = u_+\,, \tag{3g}$$

$$u u_- = -u_-\,, \tag{3h}$$

$$(u_\pm)^- = u_\mp\,. \tag{3i}$$

You should prove (3c) – (3i). By the way, these two special unipodes $u_\pm$ square to themselves. Such numbers in a ring are referred to as *idempotents*. In the unipodal numbers they have no inverses. The fact that the unipodal number system is not a field is of little concern to me. In fact, most unipodes have inverses, so long as they are not multiples of one of the idempotents. If one needs field elements, the scalars of the unipodal numbers comprise the field of complex numbers.

Two often-used results are, for complex numbers w, z (which are used to convert unipodes between the bases $\{1, u\}$ and $\{u_+, u_-\}$):

$$w + zu = (w + z)u_+ + (w - z)u_-\,, \tag{4a}$$

$$wu_+ + zu_- = \tfrac{1}{2}(w + z) + \tfrac{1}{2}(w - z)u\,. \tag{4b}$$

The unipodal algebra has two copies of the complex numbers, one for each component. In any true unipodal equation, the correponding coefficients across the equal sign are equal to each other. This is similar to equating real and imaginary components across the equal sign in the complex algebra.

When I first used the unipodal algebra to solve polynomial equations (c. 1984-5), I used the Clifford 1 algebra over the complex numbers. The '1' means one unit vector u. So, a Clifford 1 number c can be represented as

$$c = a + bu\,, \tag{5}$$

where a, b are complex numbers. Of course, u being a unit vector, then

$$u^2 = 1\,. \tag{6}$$

Now, the standard basis for this space is $\{1, u\}$ and the scalars are the complex numbers. To extract the 'scalar part' of (5), we use the selection operator $\langle \cdot \rangle$, as follows:

$$\langle\, c \,\rangle = \langle\, a + bu \,\rangle = a\,, \tag{7}$$

One can also subscript the selector with a zero for the scalar part:

$$\langle\, c\,\rangle_0 = \langle\, a + bu\,\rangle_0 = a\,, \tag{8}$$

and with a '1' for the vector part:

$$\langle\, c\,\rangle_1 = \langle\, a + bu\,\rangle_1 = bu\,. \tag{9}$$

When I adopted the name 'unipodal algebra' from a paper I cowrote with two other authors, I found a need to adopt new terminology for naming the scalar and vector parts. Just as complex numbers are composed of a real number times the unit '1' and another real number times the unit imaginary i, the unipodal numbers are composed of a complex number times the unit '1' and another complex number times the unipotent number u. The part of the unipode that does not contain the unipotent factor is called the 'complex part' of the unipode. The part that does contain the unipotent element factor is called the **uniplex part** of the unipode.

Now, before you complain that calling the scalar part of a unipode the 'complex part' is nonsense, I point out that in complex analysis, the nonimaginary part is referred to as the 'real part'. Lastly, when I say the 'uniplex part' in this series of papers, I refer only to the coefficient of the nonscarlar part, which is complex only. Thus the uniplex part of unipode $c = a + bu$ is just b. Another way to think of the uniplex part of c is to take the scalar (or complex) part of cu.

$$\langle\, c\,\rangle_1 = \langle\, cu\,\rangle = \langle\, au + b\,\rangle = b\,. \tag{10}$$

Thus, one must be careful when I report I'm taking the uniplex part of a unipode (across all the papers I've written over the years), because at times it may contain that factor of u and at other times not. But like I said: In this series it will always mean only the scalar factor of the unipotent element.

Much of the algebraic power of the unipodal algebra comes from 1) it being able to switch the presentation of a unipode between the standard basis and the idempotent basis, the latter basis being well suited for taking powers and roots.

3 The Solution

I'll begin by making the standard constructions.

$$x + y = k\,, \tag{11}$$
$$x - y = \ell\,, \tag{12}$$

At this point, I want to outline a strategy for the rest of the solution. First, since I do not intend to solve for $x + y$ by first solving for x and y individually, I want to eliminate the introduced variable ℓ as soon as possible. This is the course after that:

1) Pick the 'first unipode' that will specifically faciliate employing the given constraints (1) and (2).

2) Any function of x, y that cannot be immediately replaced by the given constraints should be replaced by some function of k.

3) Look for some function of k that can be turned into a polynomial in k that can be solve analytically for its roots.

So, how to replace ℓ by something in k perhaps? From (1), we have that

$$x^2 - y^2 = (x+y)(x-y) = k\ell = \sqrt{10}\,, \tag{13}$$

from which we get

$$\ell = \frac{\sqrt{10}}{k}\,. \tag{14}$$

It turns out that we won't need ℓ unless we want go the distance and solve for x and y individually. In that case, we can make use this equation

$$x - y = \frac{\sqrt{10}}{k}\,. \tag{15}$$

But almost certainly we'll need to deal with the expression $x^2 + y^2$. So,

$$k^2 = (x+y)^2 = x^2 + 2xy + y^2 = x^2 + y^2 + 2\sqrt{10}\,, \tag{16}$$

hence,

$$x^2 + y^2 = k^2 - 2\sqrt{10}\,, \tag{17}$$

which comforms to heuristic rule 2) above.

For our first unipode, let's try

$$\begin{aligned} b &= x^2 + y^2 u = (x^2+y^2)u_+ + (x^2-y^2)u_- \\ &= (k^2 - 2\sqrt{10})\,u_+ + \sqrt{10}u_- \\ &= \tfrac{1}{2}(k^2 - \sqrt{10}) + \tfrac{1}{2}(k^2 - 3\sqrt{10})u\,. \end{aligned} \tag{18}$$

Now, the product of the coefficients of the unipode on the top line has to equal the product of the coefficients of the unipode on the bottom line, giving us

$$x^2 y^2 = \tfrac{1}{2}(k^2 - \sqrt{10})\tfrac{1}{2}(k^2 - 3\sqrt{10})\,. \tag{19}$$

But $xy = \sqrt{10}$, hence, $x^2y^2 = 10$. Therefore, with a bit of calculation, (19) becomes

$$k^4 - 4\sqrt{10}k^2 - 10 = 0\,. \tag{20}$$

I'll let Wolframalpha.com solve for the roots. For the real roots, we get

$$k = \pm\sqrt[4]{2}\,\sqrt{5+2\sqrt{5}}\,. \tag{21}$$

And for the imaginary roots:

$$k = \pm i\sqrt[4]{2}\,\sqrt{5-2\sqrt{5}}\,. \tag{22}$$

To arrive at the form of answer that Presenter gave, do this

$$k = \pm\sqrt[4]{2}\left(\frac{\sqrt[4]{5}}{\sqrt[4]{5}}\right)\sqrt{5+2\sqrt{5}} = \pm\sqrt[4]{10}\sqrt{\sqrt{5}+2}\,. \tag{23}$$

4 Aftermath

If you want to solve for x and y individually, now that we have the values of k, use them in Equation (11) and couple that with Equation (15).