Math Diversion Problem 226

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Keep an open mind. That's the secret.
— Doctor Who

The YouTube video is found at:

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Source: https://www.youtube.com/watch?v=FLLTAHlEqLk
Title: France | Junior Math Olympiad Exponent
 Presenter: Super Academy
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1 The Problem

Given the relation

$$x^{\log 27} + 9^{\log x} = 36\,, \tag{1}$$

find the values of x.

(Skip down to the solution, if you like.)

2 Basics of Complex Numbers

Typically, we find a generic complex number denoted by the letter z, but one is free to choose other letters, as well. So, if z is a complex number, in general it has both real and imaginary parts:

$$z = a + bi\,, \tag{2}$$

where a, b are real components of basis vectors $1, i$. But they are also expressed as, respectively, the 'real' and 'imaginary' components of z.

Complex conjugation of complex number z is an operation that leaves real numbers alone but replaces the unit imaginary i with its negative, i.e., $-i$. The symbols most often used to represent complex conjugation are the $*$ and the overbar. I'll usually use the overbar. Thus, the complex conjugate of z in (2) is

$$\overline{z} = a - bi\,. \tag{3}$$

Obviously, the complex conjugation of a pure real number has no effect.

A funny thing happens when we multiply a complex number by its conjugate:

$$z\overline{z} = (a + bi)(a - bi) = a^2 + b^2 \,. \tag{4}$$

So, $z\overline{z}$ is zero if and only if $z = 0$, otherwise, it's a positive real number.

Another funny thing happens when we add a complex number and its conjugate: we also get a real number. Let's see.

$$z + \overline{z} = (a + bi) + (a - bi) = 2a \,. \tag{5}$$

Why do we care about this? Because sometimes we need to map complex numbers into the real numbers to get information on the complex numbers. This problem will show you that.

I'm not going to prove this here, but every complex number can be expressed in exponential (or polar) form:

$$z = a + bi = \sqrt{a^2 + b^2} e^{i\theta} = (z\overline{z})^{1/2} e^{i\theta} = r\, e^{i\theta} \,, \tag{6}$$

where we can think of r as the length of the complex numbers z or $\overline{z}$.

$$r \equiv (z\overline{z})^{1/2} \quad \text{or} \quad r^2 = z\overline{z} = |z|^2 \,. \tag{7}$$

So, it will be good to know all this stuff in this section before you attempt to follow my solutions to these complex variables problems.

By the way, the complex numbers are what's called a *field*, so they can be added, subtracted, multiplied, and divided by each other (except you can't divide by zero, as usual). And, therefore, you can apply the quadratic formula to them! (Yay!)

Lemma 1: If a complex number z is equal to its own conjugate $z = \overline{z}$, it's real.

Lemma 2: If a complex number z is complex conjugated twice then there's no change: $\overline{\overline{z}} = z$.

Lemma 3: The complex conjugated of a product or a sum is the product or sum of the complex conjugates: $\overline{z_1 z_2} = \overline{z}_1 \overline{z}_2$ and $\overline{z_1 + z_2} = \overline{z}_1 + \overline{z}_2$.

Lemma 4: If $s, t \in \mathbb{R}$ and $z = s + ti$ then

$$i\overline{z} = t + si \,. \tag{8}$$

3 The Solution

The Given equation has two bases: x and 9. I want only one, namely 9. Therefore, let

$$x = 9^y \,. \tag{9}$$

Then (1) becomes

$$(9^y)^{\log 27} + 9^{\log 9^y} = 36 \,, \tag{10}$$

or

$$(9^{3y})^{\log 3} + 9^{\log 3^{2y}} = 36 \,, \tag{11}$$

or

$$(9^{3y})^{\log 3} + (9^{2y})^{\log 3} = 36 \,, \tag{12}$$

or

$$(9^{y \log 3})^3 + (9^{y \log 3})^2 = 36 \,. \tag{13}$$

Now, we let

$$z = 9^{y \log 3} \,. \tag{14}$$

Then (13) becomes

$$z^3 + z^2 - 36 = 0 \,. \tag{15}$$

By inspection we could arrive at the root $z = 3$. And then by long division, we get

$$z^3 + z^2 - 36 = (z - 3)(z^2 + 4z + 12) \,. \tag{16}$$

However, the roots from the quadratic are complex, so I will ignore then.

So, we now need to back out of all these variable substitutions.

$$z = 9^{y \log 3} = 3 \,. \tag{17}$$

From this we get that

$$y = \frac{1}{\log 9} \,. \tag{18}$$

Therefore,

$$x = 9^{\frac{1}{\log 9}} \,. \tag{19}$$

A funny thing happens if we take the logarithm of both sides:

$$\log x = \frac{1}{\log 9} \log 9 = 1 \,. \tag{20}$$

Hence,

$$x = 10 \,. \tag{21}$$