

# Math Diversion Problem 233

P. Reany

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The secret to perseverance is to just keep doing it.  
— The Author

The YouTube video is found at:

Source: <https://www.youtube.com/watch?v=ls8t1LAGcBI>  
Title: Germany Math Olympiad Question  
Presenter: Higher Mathematics

## 1 The Problem

Given the relation

$$x^{\sqrt{x}} = 10, \tag{1}$$

find the values of  $x$  over the real numbers.

I have already solved this problem earlier, but this time I have a new approach to it.

## 2 Lambert $W$ Function Lemma

$$W(y \ln y) = \ln y, \tag{2}$$

for the principal value of  $W$  and  $y \ln y \geq -1/e$ .

## 3 The Solution

Let's begin by taking the square root across (1):

$$\sqrt{x}^{\sqrt{x}} = \sqrt{10}. \tag{3}$$

Next, we take the logarithm:

$$\sqrt{x} \ln \sqrt{x} = \ln \sqrt{10}. \tag{4}$$

Now we operate with the Lambert  $W$  function:

$$\ln \sqrt{x} = W(\sqrt{x} \ln \sqrt{x}) = W(\ln \sqrt{10}). \quad (5)$$

But

$$\ln \sqrt{x} = \frac{1}{2} \ln x, \quad (6)$$

so

$$\ln x = 2W\left(\frac{1}{2} \ln 10\right). \quad (7)$$

And finally,

$$x = e^{2W\left(\frac{1}{2} \ln 10\right)}. \quad (8)$$