Math Diversion Problem 236

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A clue is anything that doesn't happen
the way it oughtta happen.
— Harry Orwell, TV
show *Harry O*

The YouTube video is found at:

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Source: https://www.youtube.com/watch?v=LESu08vVvrQ
Title: International Mathematical Olympiad Problem
 Presenter: Higher Mathematics
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1 The Problem

Given the relation

$$x^5 = 9^x \,, \tag{1}$$

find the values of x over the real (or complex) numbers.

2 A Lambert W Function Lemmas

We'll start off with a very brief overview of the Lambert W function. It was invented to unravel this product in variable x:

$$xe^x = A \,, \tag{2}$$

to solve for x. What a cool function, indeed! Hence,

$$x = W(xe^x) = W(A) \,. \tag{3}$$

And that's it, at least for us here.

Now, two warnings: First, we will not attempt to reduce $W(A)$ to numeric form, unless it has a simple identity we can apply to it. Second, the issues of the domain of W can be complicated, and for the most part we will be only interested in the so-called 'principal value' of the W function, denoted as W_0, though we'll suppress the subscript zero; and the domain issue is as follows: for $W(z)$ the $\mathrm{Re}(z) \geq -1/e$.

Lemma 1:

The following is the 'Lambert W function base s'[1], or W_s, where s is a positive real number. Let's begin with the relation

$$xs^x = A\,, \tag{4}$$

which looks very similar to (2). Then

$$x = W_s(xs^x) \equiv \frac{W(A \ln s)}{\ln s}\,. \tag{5}$$

But when $s = e$, we have that

$$x = W_e(xe^x) = \frac{W(A \ln e)}{\ln e} = W(A)\,, \tag{6}$$

which is the usual Lambert W function. (By the way, the proof to this lemma is not hard. It begins with setting $s^x = e^y$ and proceeding from there.)

Lemma 2:

Let's begin with the relation

$$xe^x = A\,, \tag{7}$$

from which, as we have seen, we get the fundamental defining result:

$$x = W(xe^x) = W(A)\,. \tag{8}$$

If we now make the change of variable:

$$x = \ln y\,, \tag{9}$$

then (8) becomes

$$\ln y = W(y \ln y) = W(A)\,. \tag{10}$$

In other words, besides trying to conform the given expression to that of (7), we can alternatively conform it to this:

$$y \ln y = A\,, \tag{11}$$

with (next-step) solution

$$\ln y = W(A)\,. \tag{12}$$

So, if you are solving for x, then

$$x = \ln y = W(A)\,. \tag{13}$$

But if you're solving for y, then,

$$y = e^{W(A)}\,. \tag{14}$$

[1]This notation I invented myself.

Lemma 3:

Let's begin with relation (12) and multiply through by y, to get

$$y \ln y = yW(A)\,. \tag{15}$$

But according to (11), $y \ln y = A$, so that

$$A = yW(A)\,. \tag{16}$$

Now, from (9) we get that

$$y = e^x\,, \tag{17}$$

which, with some algebraic steps, gives us

$$e^{-x} = \frac{W(A)}{A}\,. \tag{18}$$

But from (8), we know that

$$e^{-W(A)} = \frac{W(A)}{A}\,, \tag{19}$$

or, more familiarly as,

$$W(A)e^{W(A)} = A\,. \tag{20}$$

We should recognize that these last two equations are totally general, so long as one adheres to the appropriate domain restrictions on them.

By the way, which form one gets of (18), either the LHS or the RHS (if either), is very much dependent on the path one takes through one's solution space. For example, you may get one of them, but WolframAlpha or the Presenter might get the other, and these equations are how we are to translate between them.

3 The Solution

First, I want to perform a change of variable:

$$x = 9^\alpha\,, \tag{21}$$

then (1) becomes

$$(9^\alpha)^5 = 9^{9^\alpha}\,. \tag{22}$$

On equating exponents, we get

$$5\alpha = 9^\alpha\,. \tag{23}$$

On failing to find an integral or rational solution for α, we look elsewhere. Rearranging, we get

$$\alpha 9^{-\alpha} = 5^{-1}\,, \tag{24}$$

and some more altering:

$$-\alpha 9^{-\alpha} = -5^{-1} \,. \tag{25}$$

Therefore, by the first lemma above,

$$-\alpha = W_9(-5^{-1}) = \frac{W(-5^{-1}\ln 9)}{\ln 9} \,. \tag{26}$$

Thus,

$$\alpha = -\frac{W(-5^{-1}\ln 9)}{\ln 9} \,. \tag{27}$$

The way we have proceeded, $x = 9^{\alpha}$, but we can do better. Let

$$9^{\alpha} = e^{z} \,, \tag{28}$$

where $z = \alpha \ln 9$. Therefore,

$$x = e^{z} = e^{-W(-5^{-1}\ln 9)} \,. \tag{29}$$

Using (19), we can also write this as

$$x = \frac{W(-5^{-1}\ln 9)}{-5^{-1}\ln 9} = \frac{-5W(-5^{-1}\ln 9)}{\ln 9} \,, \tag{30}$$

which, essentially, is WolframAlpha's solution. Whereas, (29) is, essentially, the Presenter's solution.