Math Diversion Problem 237

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> It is clear that the chief end of mathematical study
> must be to make the students think.
> — John Wesley Young
> (So think!)

The YouTube video is found at:

1 The Problem

Given the relation
$$a^3 + b^3 + 3ab = 1 \,, \tag{1}$$

find the values of $a + b$ over the real numbers. For convenience, let

$$\phi \equiv a + b \,. \tag{2}$$

2 The Solution

Let

$$\phi^3 - 0 = (a + b)^3 - (a^3 + b^3 + 3ab - 1) \tag{3a}$$
$$= a^3 + 3a^2b + 3ab^2 + b^3 - (a^3 + b^3 + 3ab - 1) \tag{3b}$$
$$= 3a^2b + 3ab^2 - 3ab + 1 \tag{3c}$$
$$= 3ab(b + a - 1) + 1 \tag{3d}$$
$$= 3ab(\phi - 1) + 1 \tag{3e}$$

By inspection, we can see that $\phi = 1$ is a solution.