

# Math Diversion Problem 238

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Truth, like oil, will in time rise to surface.

— Charlie Chan

The YouTube video is found at:

Source: <https://www.youtube.com/watch?v=6A5T1uhIozs>

Title: An Interesting Homemade Equation

Problem 459

Presenter: aplusbi

## 1 The Problem

Given the relation

$$z|z|^2 = 2i, \tag{1}$$

find the values of  $z$ .

Note: WolframAlpha was of no help on this one.

## 2 The Solution

Let's begin by taking the complex conjugate of (1):

$$\bar{z}|z|^2 = -2i, \tag{2}$$

and then multiplying them together, noting that  $z\bar{z} = |z|^2 = r^2$ , we get:

$$(r^2)^{r^2} = 4. \tag{3}$$

Next, we apply the logarithm:

$$(r^2) \ln(r^2) = \ln 4. \tag{4}$$

Now we apply the Lambert  $W$  function, to get<sup>1</sup>

$$\ln(r^2) = W(\ln 4). \tag{5}$$

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<sup>1</sup>See the Appendix for my lemmas on the Lambert  $W$  function.

On exponentiating, we get

$$r^2 = e^{W(\ln 4)}. \quad (6)$$

Therefore,

$$r = e^{\frac{1}{2}W(\ln 4)} \equiv r_0, \quad (7)$$

where by  $r_0$ , I mean to indicate that its value has been determined. Thus, going back to (1), we have that

$$z = (2i)^{1/r_0^2} = (2)^{1/r_0^2} \cdot i^{1/r_0^2}. \quad (8)$$

But we can write  $i$  as  $e^{i\pi/2}$ , thus

$$z = (2)^{1/r_0^2} \cdot (e^{i\pi/2})^{1/r_0^2} = (2)^{1/r_0^2} \cdot (e^{i\pi/2r_0^2}), \quad (9)$$

to represent the principal value.

### 3 Appendix: Lambert $W$ Function Lemmas

We'll start off with a very brief overview of the Lambert  $W$  function. It was invented to unravel this product in variable  $x$ :

$$xe^x = A, \tag{10}$$

to solve for  $x$ . What a cool function, indeed! Hence,

$$x = W(xe^x) = W(A). \tag{11}$$

And that's it, at least for us here.

Now, two warnings: First, we will not attempt to reduce  $W(A)$  to numeric form, unless it has a simple identity we can apply to it. Second, the issues of the domain of  $W$  can be complicated, and for the most part we will be only interested in the so-called 'principal value' of the  $W$  function, denoted as  $W_0$ , though we'll suppress the subscript zero; and the domain issue is as follows: for  $W(z)$  the  $\text{Re}(z) \geq -1/e$ .

**Lemma 1:**

The following is the 'Lambert  $W$  function base  $s$ '<sup>2</sup>, or  $W_s$ , where  $s$  is a positive real number. Let's begin with the relation

$$xs^x = A, \tag{12}$$

which looks very similar to (10). Then

$$x = W_s(xs^x) \equiv \frac{W(A \ln s)}{\ln s}. \tag{13}$$

But when  $s = e$ , we have that

$$x = W_e(xe^x) = \frac{W(A \ln e)}{\ln e} = W(A), \tag{14}$$

which is the usual Lambert  $W$  function. (By the way, the proof to this lemma is not hard. It begins with setting  $s^x = e^y$  and proceeding from there.)

**Lemma 2:**

Let's begin with the relation

$$xe^x = A, \tag{15}$$

from which, as we have seen, we get the fundamental defining result:

$$x = W(xe^x) = W(A). \tag{16}$$

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<sup>2</sup>This notation I invented myself.

If we now make the change of variable:

$$x = \ln y, \tag{17}$$

then (16) becomes

$$\ln y = W(y \ln y) = W(A). \tag{18}$$

In other words, besides trying to conform the given expression to that of (15), we can alternatively conform it to this:

$$y \ln y = A, \tag{19}$$

with (next-step) solution

$$\ln y = W(A). \tag{20}$$

So, if you are solving for  $x$ , then

$$x = \ln y = W(A). \tag{21}$$

But if you're solving for  $y$ , then,

$$y = e^{W(A)}. \tag{22}$$

**Lemma 3:**

Let's begin with relation (20) and multiply through by  $y$ , to get

$$y \ln y = yW(A). \tag{23}$$

But according to (19),  $y \ln y = A$ , so that

$$A = yW(A). \tag{24}$$

Now, from (17) we get that

$$y = e^x, \tag{25}$$

which, with some algebraic steps, gives us

$$e^{-x} = \frac{W(A)}{A}. \tag{26}$$

But from (16), we know that

$$e^{-W(A)} = \frac{W(A)}{A}, \tag{27}$$

or, more familiarly as,

$$W(A)e^{W(A)} = A. \tag{28}$$

We should recognize that these last two equations are totally general, so long as one adheres to the appropriate domain restrictions on them.

By the way, which form one gets of (26), either the LHS or the RHS (if either), is very much dependent on the path one takes through one's solution space. For example, you may get one of them, but WolframAlpha or the Presenter might get the other, and these equations are how we are to translate between them.