Math Diversion Problem 240

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Learning is a treasure that will follow its owner everywhere.
— Chinese proverb

The YouTube video is found at:

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Source: https://www.youtube.com/watch?v=Xmquv6CpTco
Title: Can We Solve A Tangential Equation
          Problem 435
 Presenter: aplusbi
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1 The Problem

Given the relation

$$(1 + i \tan\theta)^5 = 32\,, \tag{1}$$

find the values of θ.

2 The Solution

Or, should I say *my* solution.

The Given equation can be extended thusly,

$$(1 + i \tan\theta)^5 = 32 = 2^5 e^{2\pi i n} \quad \text{for } n \in \mathbb{Z}\,. \tag{2}$$

On taking the fifth root of both sides, we get

$$1 + i \tan\theta = 2e^{2\pi i n/5} \quad \text{for } n \in \mathbb{Z}\,. \tag{3}$$

For convenience, let's define

$$\alpha_n \equiv e^{2\pi i n/5}\,. \tag{4}$$

Then (6) becomes

$$1 + i \tan\theta = 2\alpha_n \quad \text{for } n \in \mathbb{Z}\,. \tag{5}$$

Now we multiply through by $\cos\theta$.

$$\cos\theta + i\sin\theta = 2\alpha_n \cos\theta \quad \text{for } n \in \mathbb{Z}\,. \tag{6}$$

But, $\cos\theta + i\sin\theta = e^{i\theta}$, so (6) becomes

$$e^{i\theta} = 2\alpha_n \cos\theta \quad \text{for } n \in \mathbb{Z}\,. \tag{7}$$

However,

$$\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2}\,, \tag{8}$$

thus (7) becomes

$$e^{i\theta} = \alpha_n(e^{i\theta} + e^{-i\theta}) \quad \text{for } n \in \mathbb{Z}\,. \tag{9}$$

From this we can get

$$e^{2i\theta} = \frac{\alpha_n}{1-\alpha_n} \quad \text{for } n \in \mathbb{Z}\,. \tag{10}$$

And finally, θ is

$$\theta = \frac{i}{2}\ln\big(\frac{1-\alpha_n}{\alpha_n}\big) \quad \text{for } n \in \mathbb{Z}\,. \tag{11}$$