Math Diversion Problem 241

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> Don't ever take a fence down until you know the reason it was put up.
> — Chesterton

The YouTube video is found at:

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Source: https://www.youtube.com/watch?v=wZSxh_WB_JY
Title: An Interesting Nonstandard Equation
 Presenter: SyberMath
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1 The Problem

Given the relation

$$e^{x^2-1} = x\,, \tag{1}$$

find the values of x.

2 The Solution

Let's begin by stating the trivial solution $x = 1$.

To continue, I intend to use the Lambert W function lemma, that, for $a > 0$, given

$$za^z = B\,, \tag{2}$$

then

$$z = W_a(B)\,, \tag{3}$$

where

$$W_a(B) \equiv \frac{W(B\ln a)}{\ln a}\,. \tag{4}$$

OK, so next we square the Given equation:[1]

$$(e^2)^{x^2-1} = x^2\,. \tag{5}$$

[1] This might introduce an extraneous root.

Then we reconfigure this equation to pattern match to the lemma.

$$e^{-2} = x^2(e^2)^{-x^2}\,. \tag{6}$$

Then take the negatives:

$$-e^{-2} = (-x^2)(e^2)^{-x^2}\,. \tag{7}$$

Now, let

$$a = e^2\,. \tag{8}$$

Then (7) becomes[2]

$$-a^{-1} = (-x^2)a^{-x^2}\,. \tag{9}$$

So, on applying the lemma, we get

$$-x^2 = W_a(-a^{-1}) = \frac{W(-a^{-1}\ln a)}{\ln a}\,. \tag{10}$$

But $\ln a = 2$. Therefore,

$$-x^2 = \frac{W(-2/e^2)}{2}\,. \tag{11}$$

Thus,

$$x = \pm i\frac{\sqrt{W(-2/e^2)}}{\sqrt{2}}\,. \tag{12}$$

I don't know how to check if the negative root is extraneous or not.

[2]The quantity $-a^{-1}$ plays to role of B in the above lemma.