Math Diversion Problem 246

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People often overlook the obvious.
— Doctor Who

The YouTube video is found at:

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Source: https://www.youtube.com/watch?v=bCG6L0h5R4A
Title: The Hardest SAT Problem
 Presenter: Higher Mathematics
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1 The Problem

Given the relation

$$x^4 = 3^x \,, \tag{1}$$

find the values of x.

2 The Preparation

I intend to use the Lambert W function lemma, that, for $a > 0$, given

$$za^z = B \,, \tag{2}$$

then

$$z = W_a(B) \,, \tag{3}$$

where

$$W_a(B) \equiv \frac{W(B \ln a)}{\ln a} \,. \tag{4}$$

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3 The Solution

I'll start by substituting $x = 3^\alpha$ into the Given equation:

$$(3^\alpha)^4 = 3^{3^\alpha}\,. \tag{5}$$

On setting the exponents equal, we have

$$4\alpha = 3^\alpha\,. \tag{6}$$

There being no rational solution to α, we turn to the Lambert W function for assistance. First, a little algebraic manipulation.

$$\alpha 3^{-\alpha} = \tfrac{1}{4}\,. \tag{7}$$

Next, we multiply through by -1:

$$-\alpha 3^{-\alpha} = -\tfrac{1}{4}\,. \tag{8}$$

So, according to the above lemma:

$$-\alpha = W_3(-\tfrac{1}{4}) = \frac{W(-\frac{1}{4}\ln 3)}{\ln 3}\,. \tag{9}$$

Hence,

$$x = 3^{-W(-\frac{1}{4}\ln 3)/\ln 3}\,. \tag{10}$$

But this can also be expressed in the natural exponential form by using the following identity: Given

$$3^Q = e^P\,, \tag{11}$$

then

$$P = Q\ln 3\,. \tag{12}$$

Therefore,

$$x = e^{-W(-\frac{1}{4}\ln 3)}\,. \tag{13}$$