Math Diversion Problem 249

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> You don't understand anything until you learn
> it more than one way.
> — Marvin Minsky

The YouTube video is found at:

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Source: https://www.youtube.com/watch?v=QkiWK7mk3bc
Title: Nice Exponent Math Simplification
 Presenter: Master T Maths Class
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1   The Problem

Given the relation

$$5^x \cdot 2^x \cdot x^x = \sqrt{5}\,, \tag{1}$$

find the values of x.

2   The Preparation

I intend to use the Lambert W function lemma, that, if

$$y \ln y = B\,, \tag{2}$$

then

$$\ln y = W(B)\,. \tag{3}$$

3   The Solution

$$5^x \cdot 2^x \cdot x^x = \sqrt{5}\,, \tag{4}$$

The LHS of the Given relation can be condensed a bit.

$$(10x)^x = \sqrt{5}\,. \tag{5}$$

by inspection, we can find the solution

$$x = 1/2 \,. \tag{6}$$

Let's find another. Raise both sides of (5) of the tenth power.

$$(10x)^{10x} = 5^5 \,. \tag{7}$$

Next, apply the logarithm.

$$(10x)\ln(10x) = 5\ln 5 \,. \tag{8}$$

Now we apply the lemma:

$$\ln(10x) = W(5\ln 5) \,, \tag{9}$$

And the result is

$$10x = e^{W(5\ln 5)} \,, \tag{10}$$

or

$$x = \frac{1}{10} e^{W(5\ln 5)} \,, \tag{11}$$

However, since $W(5\ln 5) = \ln 5$, then

$$x = \frac{1}{10} e^{\ln 5} = \frac{5}{10} = \frac{1}{2} \,. \tag{12}$$

So, to get complex solution, we need to go to $W_n(5\ln 5)$.