Math Diversion Problem 251

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> The greatest killer of creativity is interruption.
> — John Cleese

The YouTube video is found at:

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Source: https://www.youtube.com/watch?v=YknH4GYKdBo
Title: An Interesting Nonstandard Equation
 Presenter: SyberMath
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1 The Problem

Given the relation

$$\sqrt{x} = \left(\frac{1}{2}\right)^x , \tag{1}$$

find the real values of x.

2 The Preparation

I intend to use the Lambert W function, which goes as follows:

$$ze^z = B , \tag{2}$$

then

$$z = W(B) , \tag{3}$$

where there are domain constraints on B that we won't go into here. Warning: This can be a complicated (multi-valued) function to deal with.

I also intend to use the Lambert W function Lemma 1, that, for $a > 0$, given

$$za^z = B , \tag{4}$$

then

$$z = W_a(B) , \tag{5}$$

where

$$W_a(B) \equiv \frac{W(B \ln a)}{\ln a}\,, \tag{6}$$

which becomes the ordinary Lambert W function when $a = e$.

Another lemma I'll need is the identity (Lemma 2), that, if

$$y \ln y = B\,, \tag{7}$$

then

$$\ln y = W(B)\,. \tag{8}$$

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3 The Solution

Let's begin by squaring the Given equation to get

$$x = \left(\frac{1}{4}\right)^x\,. \tag{9}$$

Now, we introduce a new variable a, defined by

$$a = \tfrac{1}{4}\,, \tag{10}$$

so that (9) becomes (with some algebra)

$$xa^{-x} = 1\,, \tag{11}$$

or

$$-xa^{-x} = -1\,. \tag{12}$$

We're now setup to use Lemma 1:

$$-x = W_a(-1) = \frac{W(-1(\ln a))}{\ln a}\,. \tag{13}$$

But the $\ln a = \ln(\frac{1}{4}) = -2\ln 2$, therefore, we get

$$-x = \frac{W(-1(-2\ln 2))}{(-2\ln 2)} = -\frac{W(2\ln 2)}{2\ln 2}\,. \tag{14}$$

On multiplying through by -1 and using Lemma 2, we have that

$$x = \frac{\ln 2}{2\ln 2} = \frac{1}{2}\,. \tag{15}$$

Indeed, this does saisfy (1).