Math Diversions, Problem 3

P. Reany

September 13, 2022

1 Problem

This problem is found on the YouTube channel **Math Window**, from September 7, 2022. My solution here is a little different from that given by the presenter.

Statement of the problem:

Given the system of equations

$$a + b = abc\,, \tag{1}$$
$$b + c = abc\,, \tag{2}$$
$$c + a = abc\,, \tag{3}$$

find a nontrivial solution.

Solution to the problem:

Of course, there's the trivial solution $a = b = c = 0$. I suggested that there is an approach to this problem using symmetry. By symmetry, I mean some form of invariance of the equations under exchange of variables. Now, it's obvious the none of the equations by themselves is invariant under every exchange of variables. For instance, Eq. (1) changes form when we exchange a and c.

However, it's evident that the system of equations as a whole is invariant under the swapping of any pair of variables. Hence, we must conclude that, if there is a nontrivial solution to the system, it must occur when $a = b = c$. Under this assumption, we can use any one of the equations to get, for instance,

$$2a = a^3\,, \tag{4}$$

which has solutions

$$a = \pm\sqrt{2}\,. \tag{5}$$

And, of course, we also have that

$$b = \pm\sqrt{2} \quad \text{and} \quad c = \pm\sqrt{2}\,. \tag{6}$$

However, there is the restriction on the solutions that the signs are either all positive or all negative for the nontrivial solutions.

But wait, there's more! There's another method that uses symmetry to get the answers. This approach begins with the following question: Can I reduce a system of equations in many variables into just one equation that is symmetric in all its variables?

It turns out that in this problem we can. If we just add the three equations together, we get

$$2(a+b+c) = 3abc\,. \tag{7}$$

This equation is clearly symmetric under the interchange of any pair of variables, hence, it must have a solution when all variables are set equal to each other, giving us the equation, for instance,

$$6a = 3a^3\,, \tag{8}$$

which can be simplified to

$$2 = a^2\,, \tag{9}$$

which is what we got before. And so we obtain the same solutions as before.