Math Diversions, Problem 33

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People often overlook the obvious. — Doctor Who

1 Problem

This problem was inspired by recent 'olympiad' problems.

Perform the indefinite integral.

$$I = \int \ln\,(x + \sqrt{1+x^2})\,dx\,. \tag{1}$$

2 Solution

If the integral had been instead

$$\int \ln\,(x + \sqrt{1-x^2})\,dx\,, \tag{2}$$

one might be tempted to try a change of variable to circular trigonometric functions, but as it is, to what? Perhaps to hyperbolic functions? Let's try it!

The fundamental relationships we need are these:

$$\cosh^2 y - \sinh^2 y = 1\,, \tag{3a}$$

$$\cosh y + \sinh y = e^y\,, \tag{3b}$$

$$\frac{d}{dy}\cosh y = \sinh y\,, \tag{3c}$$

$$\frac{d}{dy}\sinh y = \cosh y\,. \tag{3d}$$

If we let

$$x = \sinh y \quad \text{then} \quad \sqrt{1+x^2} = \cosh y\,. \tag{4}$$

Therefore

$$x + \sqrt{1+x^2} = \sinh y + \cosh y = e^y\,. \tag{5}$$

On substituting this into (1), we get

$$\int y\, dx\,. \tag{6}$$

Next, we need to convert the differential dx. Differentiating $x = \sinh y$, we have that

$$dx = \cosh y\, dy\,. \tag{7}$$

On substituting into (6), we get

$$\int y \cosh y\, dy = y \sinh y - \cosh y + c\,, \tag{8}$$

where c is an arbitrary constant. Going back into x, we get

$$I = (\sinh^{-1} x)x - \sqrt{1+x^2} + c\,, \tag{9}$$

which can also be expressed as

$$I = x \ln(x + \sqrt{1+x^2}) - \sqrt{1+x^2} + c\,. \tag{10}$$

3 Afterglow

There are two great things about the hyperbolic functions:

1) The $\cosh x$ function is always positive.

2) The $\sinh x$ function is always invertible.

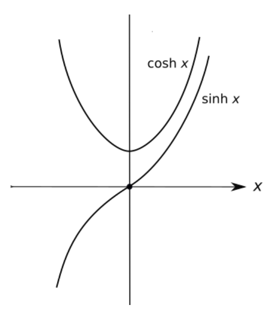

Figure 1. The hyperbolic cosine and sine when x is real valued.