Math Diversions, Problem 37

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People often overlook the obvious. — Doctor Who

1 Problem

The YouTube video is found at:

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https://www.youtube.com/watch?v=Srn-PJwFZgg
  Titled: A fun proof for an integerl
  Presenter: Prime Newtons
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If n is a positive integer, show that[1]

$$\frac{n}{6} + \frac{n^2}{2} + \frac{n^3}{3} \tag{1}$$

is also a positive integer.

2 Solution

Proof by induction.

First, we establish that the expression is a positive integer for the base case of $n = 1$.

$$\frac{1}{6} + \frac{1}{2} + \frac{1}{3} = 1\,, \tag{2}$$

and 1 is a positive integer.

This is where we use the inductive hypothesis: For some arbitrary positive integer n, let

$$a \equiv \frac{n}{6} + \frac{n^2}{2} + \frac{n^3}{3}\,, \tag{3}$$

where a is some integer. Now, assuming this last equation is true, we have to show that it remains true when $n \to n+1$. As a mere visual aide, I'm setting

[1] I changed the problem slightly.

the new expression equal to x (which is quite likely not equal to a). Thus,

$$\begin{aligned} x &= \frac{n+1}{6} + \frac{(n+1)^2}{2} + \frac{(n+1)^3}{3} \\ &= \frac{n+1}{6} + \frac{n^2+2n+1}{2} + \frac{n^3+3n^2+3n+1}{3} \\ &= \left(\frac{1}{6} + \frac{1}{2} + \frac{1}{3}\right) + \frac{n}{6} + \frac{n^2+2n}{2} + \frac{n^3+3n^2+3n}{3} \\ &= 1 + \left(\frac{n}{6} + \frac{n^2}{2} + \frac{n^3}{3}\right) + n + n^2 + n \\ &= 1 + a + 2n + n^2 \,. \end{aligned} \tag{4}$$

Thus x, being the sum of positive integers, is itself a positive integer, and we are finished.