Math Diversions, Problem 38

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People often overlook the obvious.
— Doctor Who

1 Problem

The YouTube video is found at:

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https://www.youtube.com/watch?v=_jaPL00JUhU
   Titled: Can You Simplify A Radical?
   Presenter: SyberMath
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Simplify the following radical.

$$R = \sqrt{a + \sqrt{a^2 - 1}}\,. \tag{1}$$

2 Solution

Let's again attempt a solution using the hyperbolic functions, which are described briefly below:

The fundamental relationships we need are these:

$$\cosh^2 y - \sinh^2 y = 1\,, \tag{2a}$$
$$\cosh y + \sinh y = e^{y}\,, \tag{2b}$$
$$\cosh y - \sinh y = e^{-y}\,, \tag{2c}$$
$$\cosh \tfrac{1}{2}y = \sqrt{\frac{\cosh y + 1}{2}}\,, \tag{2d}$$
$$\sinh \tfrac{1}{2}y = \operatorname{sgn}(y)\sqrt{\frac{\cosh y - 1}{2}}\,. \tag{2e}$$

If we let

$$a = \cosh y\,, \tag{3}$$

Then

$$\sqrt{a^2 - 1} = \sinh y\,. \tag{4}$$

Then

$$R = \sqrt{\cosh x + \sinh y} = \sqrt{e^y} = e^{\frac{1}{2}y} \tag{5}$$

But

$$\begin{aligned} e^{\frac{1}{2}y} &= \cosh \tfrac{1}{2}y + \sinh \tfrac{1}{2}y \\ &= \sqrt{\frac{\cosh y + 1}{2}} + \operatorname{sgn}(y)\sqrt{\frac{\cosh y - 1}{2}} \end{aligned} \tag{6}$$

Hence,

$$R = \sqrt{\frac{a+1}{2}} + \sqrt{\frac{a-1}{2}}\,, \tag{7}$$

where we have restricted y to the nonnegative real numbers.