

Math Diversions, Problem 43

P. Reany

September 16, 2024

People often overlook the obvious.
— Doctor Who

1 Problem

The YouTube video is found at:

<https://www.youtube.com/watch?v=9v41Nxu9UMU>
Titled: Harvard University Exponential Problem
Presenter: Super Academy

Given the relation

$$2^x \cdot 3^{x^2} = 6, \tag{1}$$

find the values of x .

2 Solution

There are so many ways to approach the solution to this problem. Mine is just one of them. Our first job is to factor 6.

$$2^x \cdot 3^{x^2} = 2 \cdot 3. \tag{2}$$

We can now recast the original equation to

$$2^{x-1} \cdot 3^{x^2-1} = 1. \tag{3}$$

The way forward is either to replace the base 2 by a power of 3, or replace 3 by a power of 2. The former looks better.

$$2 = 3^a. \tag{4}$$

Perhaps you wonder how I know that I can even do this. Well, if you believe in the real numbers and in logarithms, you have to believe there exists an a that

makes (4) true. If you don't believe in the real numbers, you are going to have a hard time in accepted mathematics. According to the rules of logarithms,

$$a = \frac{1}{\log_2 3}. \quad (5)$$

Now, (3) becomes

$$3^{a(x-1)} \cdot 3^{x^2-1} = 3^{a(x-1)+x^2-1} = 1. \quad (6)$$

So, we have declare the exponent as vanishing.

$$x^2 + ax - (a + 1) = 0. \quad (7)$$

And so its roots are:

$$x = \frac{-a \pm \sqrt{a^2 + 4(a + 1)}}{2} \quad (8a)$$

$$= \frac{-a \pm \sqrt{(a + 2)^2}}{2} \quad (8b)$$

$$= \frac{-a \pm (a + 2)}{2}. \quad (8c)$$

Therefore

$$x_+ = 1 \quad (9a)$$

$$x_- = -a - 1 = -\frac{1}{\log_2 3} - 1. \quad (9b)$$