Math Diversions, Problem 52

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> People often overlook the obvious.
> — Doctor Who

1 Problem

The YouTube video is found at:

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Source: https://www.youtube.com/watch?v=8OxLe4BDJbU
 Title: Maths Olympiad | How To Solve Olympiad Maths faster
  Presenter: Maths Atoka
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Given the relation

$$(x + \sqrt{1 + x^2})(y + \sqrt{1 + y^2}) = 1 \,, \tag{1}$$

find the values of $(x + y)^2$.[1]

2 Solution

Let's take the natural logarithm on both sides:

$$\ln(x + \sqrt{1 + x^2}) + \ln(y + \sqrt{1 + y^2}) = 0 \,. \tag{2}$$

Seems to me that the next step should be to transform to inverse sinh functions.

$$\sinh^{-1} x + \sinh^{-1} y = 0 \,. \tag{3}$$

Hence,

$$\sinh^{-1} x = -\sinh^{-1} y = \sinh^{-1}(-y) \,. \tag{4}$$

But the sinh function has an inverse over all of its range. Therefore,

$$x = -y \,, \tag{5}$$

with the conclusion that

$$(x + y)^2 = 0 \,. \tag{6}$$

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[1] According to Wolframalpha.com, the answer is zero.