Math Diversions, Problem 5

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1 Problem

This problem is found on the YouTube channel **Ankit Math Magics**, from April 14, 2023:

`https://www.youtube.com/watch?v=Y-dAhOmR_68`

My solution here is a little different from that given by the presenter:

Statement of the problem:

Given the following equation, solve for x:

$$3^x \cdot 7^{x^2} = 21\,. \tag{1}$$

The presenter gave a solution along this line: Take the log across the equation, resulting in

$$x \log 3 + x^2 \log 7 = \log 21\,, \tag{2}$$

where the base of the logarithm is not too important. Anyway, the presenter then rewrote this equation in standard form to apply the quadratic formula, which he did.[1] By the way, by the Fundamental Theorem of Algebra we know that (2) has two solutions for x.

2 My Solution

Now, in my case, as soon as I saw this equation, it looked to me to try something very different. Since $21 = 3 \cdot 7$, I rewrote (1) to get

$$3^x \cdot 7^{x^2} = 3 \cdot 7\,. \tag{3}$$

Then, it was obvious that one solution is $x = 1$, giving us:

$$3^1 \cdot 7^1 = 3 \cdot 7\,. \tag{4}$$

[1]This seems the most general way to proceed with this kind of problem.

Then it occurred to me to divide (1) by (4) to get:

$$3^{x-1} \cdot 7^{x^2-1} = 1\,. \tag{5}$$

Having unity as the RHS is a rather beneficial constraint, and here's why. Rewrite (5) in the form

$$[\,3 \cdot 7^{x+1}\,]^{x-1} = 1\,. \tag{6}$$

Once again, we see that $x = 1$ is a solution. But is there another? Requiring $x \neq 1$ forces a constraint on what's inside the square brackets, namely,

$$3 \cdot 7^{x+1} = 1\,, \tag{7}$$

which is a much simpler equation to deal with than (1). Now it's time to take a logarithm, but this time by base 7,[2] to get:

$$\log_7 3 + (x+1) = 0\,. \tag{8}$$

Solving for x, we get the second solution:

$$x = -1 - \log_7 3\,. \tag{9}$$

3 The numerical value

I asked ChatGPT to find the numerical value of $-1 - \log_7 3$ and got back ≈ -1.565.[3] Under the same request, BingChat returned ≈ -1.43.

4 Conclusion

The method used by the presenter is straightforward and algorithmic. If the problem has a solution, this method will find it. However, for a subset of problems of this type, such as the current one, a more elegant solution may be possible.

[2] Once again, which base we use is not all that important.
[3] This value is very near to a value posted by one of the reviewers of the video.