Math Diversion Problem 72

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'Obvious' is the most dangerous word in mathematics.
— Eric Temple Bell

The YouTube video is found at:

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Source: https://www.youtube.com/watch?v=QnBd2aIVEcQ
Title: A very tricky Question from Stanford
      University Entrance Exam
 Presenter: Super Academy
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1 The Problem

Given the relation

$$a^2 + 2ab + b = 44\,, \tag{1}$$

find the positive integer values of a, b. I will add to this the constraint that

$$a + b \leq 10\,. \tag{2}$$

2 The Solution

Since the allowed solution values are only positive integers, we can attempt the solution by exhaustive search. We should form a table of values, but what equation should we use? I thought it might be a bit more efficient if I reconfigured the given equation to this equivalent equation

$$a^2 + 2ab + b^2 = 44 + b^2 - b\,, \tag{3}$$

which can be reconfigured to

$$(a + b)^2 = 44 + b(b - 1)\,. \tag{4}$$

Then, if we introduce the variable n as

$$n \equiv a + b\,, \tag{5}$$

we can finalize our equation down to

$$n^2 = 44 + b(b - 1)\,. \tag{6}$$

$a+b=n$	n^2		b	$44+b(b-1)$
1	1		3	$44+3\cdot 2=50$
2	4		4	$44+4\cdot 3=56$
3	9		5	$44+5\cdot 4=64$
4	16		6	$44+6\cdot 5=74$
5	25		7	$44+7\cdot 6=86$
6	36		8	$44+8\cdot 7=100$
7	49		9	—
8	64		10	—
9	81		—	—
10	100		—	—

Table 1. We are looking for values in the column under n^2 that equal values in the fourth column. Because I imposed the condition that $n=a+b\leq 10$ then $n^2\leq 100$. Also, we must have that $b\leq 9$.

In Table 1. we see that there are two perfect squares in the fourth column: in rows 3 and 6.[1] In row 3, the perfect square is 64 with a b value of 5. In row 6, the perfect square is 100 with a b value of 8. Since $8^2=64$, the a value that goes with $b=5$ is $8-5=3$. Since $10^2=100$, the a value that goes with $b=8$ is $10-8=2$.

Hence the two solutions are

$$(a_1,b_1)=(3,5)\quad\text{and}\quad(a_2,b_2)=(2,8)\,. \tag{7}$$

[1]Technically, the first row isn't legal because it would force either a or b to be zero, which it can't be. But I included it for dual use as a row number.