Math Diversion Problem 75

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> Everything should be made as simple as
> possible, but no simpler.
> — Albert Einstein

The YouTube video is found at:

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Source: https://www.youtube.com/watch?v=QnBd2aIVEcQ
Title: A very tricky Question from Stanford
      University Entrance Exam
 Presenter: Super Academy
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1 The Problem

Given the relation

$$x - 3\sqrt{x} = 1\,, \tag{1}$$

find the value of the objective function

$$x^2 + \frac{1}{x^2}\,. \tag{2}$$

2 The Solution

Let's give our objective function a better name than 'that whatchamathing'.

$$Y \equiv x^2 + \frac{1}{x^2}\,. \tag{3}$$

Next, we 'clear of fractions':

$$x^4 + 1 = Yx^2\,. \tag{4}$$

Now, we return to the given constraint, re-arrange it, and square it, yielding

$$x^2 - 2x + 1 = 9x\,. \tag{5}$$

So, we could solve this equation for x and then plug that value/s into the objective function, but that is taking the long route. Instead, let's rewrite it to this form:[1]

$$x^2 = 11x - 1\,, \tag{6}$$

for the purpose of substituting out x^2 from (4). Hence,

$$(11x-1)^2 + 1 = Y(11x-1)\,. \tag{7}$$

Expanding the LHS:

$$121x^2 - 2(11x-1) = Y(11x-1)\,. \tag{8}$$

Substituting again:

$$121(11x-1) - 2(11x-1) = Y(11x-1)\,. \tag{9}$$

On canceling the common factor and simplifying, we have that

$$Y = 119\,. \tag{10}$$

3 Some Afterthoughts

I want to make some comments and/or suggestions that have been inferred from this problem, though they may not apply to it exactly. So, I'm thinking of some general heuristic principles.

Warning!: The heuristic principles I espouse may not be optimal for test taking, either in class or on entrance exams or on olympiad-style tests.

So, here we go:

If it's not required to solve for a particular variable, maybe it's best not to. For example, solving a quadratic equation using the quadratic formula, when it's not necessary, has the risks of introducing calculational errors, and the introducing of extraneous roots.

Avoid the reproduction of complex expressions in multiple places in your presentation. Just assign it a one-letter name and use that instead. At some point you may need to substitute the complex expression back into the system, but in this problem, I didn't. The more unnecessarily complicated an expression or equation looks, the harder it is for me to stay mindful of the big picture. When possible, simplify the look of the equations so that you don't get distracted by 'irrelevant' details. Well, don't misunderstand me: Which details are relevant or irrelevant depends on the stage of the solution you are in.

[1] I think of this relation as a 'step-down' relation. In every place we find a quadratic in x, we can replace it with a linear function of x.