Math Diversion Problem 79

P. Reany

October 9, 2024

Mathematics knows no races or geographic boundaries;
for mathematics, the cultural world is one country.
— David Hilbert

The YouTube video is found at:

```
Source: https://www.youtube.com/watch?v=Aao4AhD9q3A
Title: Mastering The Oxford University Entrance Exam
         With These Easy Tricks
 Presenter: Super Academy
```

1 The Problem

Given the relation

$$\frac{5}{x}\frac{5}{x} = \frac{x}{5}\,, \tag{1}$$

find the values of x.

2 The Solution

On reorganizing, we get

$$x^3 = 5^3\,. \tag{2}$$

Now, if we were only dealing with real numbers, the answer would be

$$x = 5\,. \tag{3}$$

However, we have to answer questions like we have been given in the complex numbers. In the complex numbers, every nth degree polynomial equation has n roots.[1] These roots are particularly easy to find when we take the n roots of unity. So, how does that happen? Simple. We start with the equation

$$x^n = 1\,, \tag{4}$$

[1] That's from the Fundamental Theorem of Algebra

and solve for x, to get

$$x = 1^{1/n}\,. \tag{5}$$

Now, we'll soon see that there are n roots contained in this equation. We demonstrate this by replacing unity with its complex form

$$1 = e^{2\pi i}\,. \tag{6}$$

Then, (5) becomes

$$x = (e^{2\pi i})^{1/n}\,. \tag{7}$$

So, when we bring the $1/n$ into the parentheses, we have to account for the fact that there are n roots of unity. This is how the mathematicians showed us how to do it:

$$x = e^{2\pi i k/n} \quad \text{where} \quad k = 0, 1, 2, \ldots, n-1\,. \tag{8}$$

There is a distinct root for each k going from 0 to $n-1$. Ta Da![2]

For the case $n = 3$, we get for the roots of unity:

$$x = e^{2\pi i k/3} \quad \text{where} \quad k = 0, 1, 2\,. \tag{9}$$

Let's look at each root.

$$x_0 = e^0 = 1\,, \quad x_1 = e^{2\pi i/3} = \tfrac{1}{2}(-1+\sqrt{3}i)\,, \quad x_2 = e^{4\pi i/3} = \tfrac{1}{2}(-1-\sqrt{3}i)\,. \tag{10}$$

So, on taking the cube root on both sides of (2), we get

$$x = 5(e^{2\pi i})^{1/3} = 5e^{2\pi i k/3} \quad \text{where} \quad k = 0, 1, 2\,, \tag{11}$$

where we used that $5 = 5e^{2\pi i}$ for our purpose of the extraction of roots.

We can now list the roots:

$$x_0 = 5\,, \quad x_1 = \frac{5}{2}(-1+\sqrt{3}i)\,, \quad x_2 = \frac{5}{2}(-1-\sqrt{3}i)\,. \tag{12}$$

3 Mathematica Notes

In Mathematica, use

```
Reduce[x^3==5^3,x]
```

to get:

$$x == 5 \;||\; x == (5(-1 - I\sqrt{3}\,))/2 \;||\; x == (5(-1 + I\sqrt{3}\,))/2 \tag{13}$$

In WolframAlpha.com that command will get you the same result, but only in the plain text version of the output (and you have to request it and then copy it to your clipboard).

[2] I won't go into this now, but the trick is to learn how to visualize these n ponts on the unit circle in the complex plane.