# Math Diversion Problem 81

P. Reany

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People often overlook the obvious.
— Doctor Who

The YouTube video is found at:

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Source: https://www.youtube.com/watch?v=KbELy5Crhf8
Title: Harvard University Admission Exam
       || Logarithms Problem Tricks
 Presenter: Super Academy
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## 1 The Problem

Given the relation

$$(x-5)^{\log(5x-25)} = 2\,, \tag{1}$$

find the values of $x$.

WolframAlpha.com claims the solutions for $x$ are

$$x = 5 + e^{\frac{1}{2}\left(\pm\sqrt{\log^2(5)+4\log(2)}-\log(5)\right)}\,, \tag{2}$$

where the logarithm is assumed to be natural.

## 2 The Solution

Let's begin with a variable substitution:

$$y \equiv x - 5\,. \tag{3}$$

Then (1) becomes

$$y^{\log(5y)} = 2\,. \tag{4}$$

Next, we take the log of both sides:

$$\log y \log(5y) = \log y(\log 5 + \log y) = \log 2\,. \tag{5}$$

Next, let's simplify:

$$\alpha \equiv \log 2\,, \quad \beta \equiv \log 5\,, \quad z \equiv \log y\,, \tag{6}$$

then (5) becomes

$$z(\beta + z) = \alpha\,, \tag{7}$$

or

$$z^2 + \beta z - \alpha = 0\,. \tag{8}$$

From this we get

$$z = \frac{-\beta \pm \sqrt{\beta^2 + 4\alpha}}{2}\,. \tag{9}$$

Next, we back up to $y$, with $y = e^z$:[1]

$$y = \exp\left\{\frac{-\beta \pm \sqrt{\beta^2 + 4\alpha}}{2}\right\}\,. \tag{10}$$

Now, we back up to $x$:

$$x = \exp\left\{\frac{-\log 5 \pm \sqrt{(\log 5)^2 + 4\log 2}}{2}\right\} + 5\,, \tag{11}$$

which can also be expressed as

$$x = \exp\left\{\frac{-\log 5 \pm \sqrt{\log^2 5 + 4\log 2}}{2}\right\} + 5\,. \tag{12}$$

If you prefer the answer in base 10:

$$x = 10^{\frac{1}{2}\left[-\log 5 \pm \sqrt{\log^2 5 + 4\log 2}\right]} + 5\,. \tag{13}$$

I leave it to the reader to further simplify this result.

Perhaps this will be helpful:

$$\log_{10} 5 = \log_{10}(10/2) = \log_{10} 10 - \log_{10} 2 = 1 - \log_{10} 2\,, \tag{14}$$

expecially under the radical sign.

Moral of the story: Don't be afraid to make variable substitutions and parameter introductions.

[1] Up to this point we haven't specifed the base to the logarithm because it wasn't necessary. But from this point on, we're declaring the base to be $e$.