Math Diversion Problem 83

P. Reany

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> I love it when a plan comes together.
> — Hannibal Smith, *The A-Team*

The YouTube video is found at:

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Source: https://www.youtube.com/watch?v=lI8DAgFfh9c
Title: Math question for a "true" geniuses
 Presenter:  Higher Mathematics
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1 The Problem

Given the relation

$$x^{\sqrt{x}} = 3\,, \tag{1}$$

find the values of x.

2 The Solution

One of many ways to start this problem is the take the square root on both sides of (1).

$$(x^{\sqrt{x}})^{\frac{1}{2}} = 3^{\frac{1}{2}}\,, \tag{2}$$

and this becomes

$$\sqrt{x}^{\sqrt{x}} = \sqrt{3}\,, \tag{3}$$

Next, a variable substitution: $y = \sqrt{x}$, which brings us to

$$y^y = \sqrt{3}\,. \tag{4}$$

Now we take the log of both sides:

$$y \log y = \log \sqrt{3}\,, \tag{5}$$

If it seems like we're getting close to using the Lambert W function, you're right. But we need another variable substitution: $y = e^z$, which brings us to

$$e^z z = \log \sqrt{3}\,, \tag{6}$$

Now, the Lambert W function is characterized by the relation

$$W(ze^z) = z\,. \tag{7}$$

So, from the last two equations, we have that

$$z = W(\log\sqrt{3})\,. \tag{8}$$

Next, we back up to y:

$$y = \exp\{W(\log\sqrt{3})\}\,. \tag{9}$$

Lastly, we back up to $x = y^2$:

$$x = \exp\{2W(\log\sqrt{3})\}\,, \tag{10}$$

which is the solution over the real numbers.