Math Diversion Problem 87

P. Reany

October 12, 2024

> The essence of mathematics lies in its freedom.
> — Georg Cantor

The YouTube video is found at:

```
Source: https://www.youtube.com/watch?v=nuy6sQckL24
Title: A beautiful Question from Harvard University Entrance Exam
 Presenter: Super Academy
```

1   The Problem

Given the relation

$$\sqrt{2}^{\sqrt{m}} - \sqrt{2}^{\sqrt{n}} = 32,768 \,, \tag{1}$$

find the values of m, n as integers.

2   The Solution

Clearly, it's best to begin with a couple variable substitutions into (1). Let

$$x = \tfrac{1}{2}\sqrt{m} \,, \tag{2a}$$
$$y = \tfrac{1}{2}\sqrt{n} \,. \tag{2b}$$

Then, the given equation becomes

$$2^x - 2^y = 32,768 \,, \tag{3}$$

where we must assume that $x > y$. As for that integer given to us:

$$32,768 = 2^{15} \,. \tag{4}$$

Thus,

$$2^x - 2^y = 2^{15} \,, \tag{5}$$

and so

$$2^{x-15} - 2^{y-15} = 1 \,. \tag{6}$$

1

Now we have to force
$$2^{y-15} = 1\,,\tag{7}$$
which forces us to set
$$y = 15 \longrightarrow n = 4y^2 = 900\ \checkmark\ .\tag{8}$$
On combining (6) and (7), we have that
$$2^{x-15} = 2\,.\tag{9}$$
Thus,
$$2^{x-16} = 1\,.\tag{10}$$
This forces us to set
$$x = 16 \longrightarrow m = 4x^2 = 4\cdot256 = 1024\ \checkmark\ .\tag{11}$$