Math Diversion Problem 88

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You cannot ask us to take sides against arithmetic.
— Winston Churchill

The YouTube video is found at:

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Source: https://www.youtube.com/watch?v=Ok_rPUyimZc
Title: Math Olympiad - Exponential Trigonometric
      Problem - find x!
 Presenter: Math Master TV
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1 The Problem

Given the relation

$$81^{\sin^2 x} + 81^{\cos^2 x} = 30\,, \tag{1}$$

find the values of x in (radians) between 0 and 2π.[1]

2 The Solution

$$81^{1-\cos^2 x} + 81^{\cos^2 x} = 30\,, \tag{2}$$

Let $a = 81$ and $b = 30$. Then (2) becomes

$$aa^{-\cos^2 x} + a^{\cos^2 x} = b\,, \tag{3}$$

or

$$aa^{-y} + a^y = b\,, \tag{4}$$

where $y = \cos^2 x$. Let $z = a^y$, then

$$az^{-1} + z = b\,, \tag{5}$$

Putting it into conventional form:

$$z^2 - bz + a = 0\,. \tag{6}$$

[1] I've change the problem slightly with an added constraint.

Solving for z:

$$
\begin{aligned}
z &= \frac{b \pm \sqrt{b^2 - 4a}}{2} && \text{(7a)}\\
&= \frac{30 \pm \sqrt{900 - 324}}{2} && \text{(7b)}\\
&= 15 \pm \sqrt{144} && \text{(7c)}\\
&= 15 \pm 12 && \text{(7d)}\\
&= 27,\ 3\,. && \text{(7e)}
\end{aligned}
$$

Then[2]

$$y = \log_{81} 27\,,\ \ y = \log_{81} 3\,. \tag{9}$$

Or, after converting the base to base 3:

$$y = \begin{cases} \log_{81} 27 & = \dfrac{\log_3 27}{\log_3 81} = \dfrac{3}{4}\,, \\ \log_{81} 3 & = \dfrac{\log_3 3}{\log_3 81} = \dfrac{1}{4}\,. \end{cases} \tag{10}$$

Since $\dfrac{3}{4} + \dfrac{1}{4} = 1$, we're probably on the right tract.

Since $y = \cos^2 x$,

$$\cos^2 x = \begin{cases} \frac{3}{4}\,, \\ \frac{1}{4}\,, \end{cases} \tag{11}$$

then

$$\cos x = \begin{cases} \pm\dfrac{\sqrt{3}}{2}\,, \\ \pm\dfrac{1}{2}\,. \end{cases} \tag{12}$$

The values that x can take on are

$$\pi/6,\ \pi/3,\ 2\pi/3,\ 5\pi/6,\ 11\pi/6,\ 5\pi/3,\ 4\pi/3,\ 7\pi/6\,. \tag{13}$$

[2] No natural logarithms this time. Use the rule

$$\log_\alpha \beta = \frac{\log_c \beta}{\log_c \alpha}\,. \tag{8}$$