Math Diversion Problem 89

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> He who would ride two camels, finds he can ride neither.
> — From an old movie

The YouTube video is found at:

1 The Problem

Given the relation

$$\sqrt{1 + \sqrt{1 + x}} = \sqrt[3]{x}\,, \tag{1}$$

find the values of x.

2 The Solution

There's no easy route out of this pit of radicals. Let

$$a \equiv \sqrt{1 + x}, \quad \text{then} \quad a^2 = 1 + x\,. \tag{2}$$

So, (1) becomes

$$\sqrt{1 + a} = \sqrt[3]{a^2 - 1}\,. \tag{3}$$

To get out of this pit, we raise both sides to the 6th power:

$$(1 + a)^3 = (a^2 - 1)^2\,, \tag{4}$$

which expands to

$$a(a^3 - a^2 - 5a - 3) = 0\,. \tag{5}$$

The possible root $a = 0$ does not work. This leaves us with

$$a^3 - a^2 - 5a - 3 = 0\,. \tag{6}$$

This has a single root at $a = 3$ and a double root at $a = -1$. This latter root does not give us a root for (1). For $a = 3$, $x = 8$, which does work.