Math Diversion Problem 93

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> A clue is anything that doesn't happen
> the way it oughtta happen.
> — Harry Orwell, TV
> show *Harry O*

The YouTube video is found at:

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Source: https://www.youtube.com/watch?v=X5RhYu88EYI
Title: Mexico | A Nice Algebra Problem
 Presenter: SALogic
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1   The Problem

Find the value of the following expression:

$$Q = \frac{(1+i)^{2024}}{(1-i)^{2023}} \,. \tag{1}$$

2   The Solution

Lemma 1:

$$\frac{(1+i)}{(1-i)} = i \,. \tag{2}$$

Proof:

$$\frac{(1+i)}{(1-i)} = \frac{(1+i)/\sqrt{2}}{(1-i)/\sqrt{2}} = \frac{e^{i\pi/4}}{e^{-i\pi/4}} = e^{i\pi/2} = i \,. \tag{3}$$

Lemma 2:

$$i^r = \begin{cases} 1 & \text{if } r = 0 \,, \\ i & \text{if } r = 1 \,, \\ -1 & \text{if } r = 2 \,, \\ -i & \text{if } r = 3 \,. \end{cases} \tag{4}$$

Lemma 3:

$$i^{4n} = 1 \qquad (5)$$

for all integers n.

Our strategy, then, will be to divide a big exponent M of i by 4 and let the remainder decide the value of the expression i^M.

Now, given that 2024 is evenly divisible by 4, the remainder after division is zero, then

$$\frac{(1+i)^{2024}}{(1-i)^{2023}} = \frac{(1+i)^{2024}}{(1-i)^{2024}}(1-i) = \left(\frac{1+i}{1-i}\right)^{2024}(1-i) = i^{2024}(1-i) = 1-i. \quad (6)$$