Math Diversion Problem 99

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October 17, 2024

Once you learn patience, your options suddenly expand.
— Robert Greene

The YouTube video is found at:

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Source: https://www.youtube.com/watch?v=t6Vzq4DnQH8
Title: A tricky Stanford University Admission Algebra Interview
 Presenter: Super Academy
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1 The Problem

Given the relation

$$\left(\frac{2}{3}\right)^x + \left(\frac{3}{2}\right)^x = 4\,, \tag{1}$$

find the values of x.

2 The Solution

There are many ways to continue from here. I chose this way. Let's change the given relation to this form

$$a^x + a^{-x} = 4\,, \tag{2}$$

where

$$a = \tfrac{2}{3}\,. \tag{3}$$

Now, we make the following substitution:

$$e^y = a^x\,. \tag{4}$$

So then (2) becomes

$$\frac{e^y + e^{-y}}{2} = 2\,, \tag{5}$$

where

$$y = x \ln a\,. \tag{6}$$

However, given that

$$\frac{e^y + e^{-y}}{2} \equiv \cosh y\,, \tag{7}$$

we can declare that

$$\cosh y = 2\,. \tag{8}$$

Hence,[1]

$$y = \pm \cosh^{-1} 2\,. \tag{9}$$

Next, we need an identity.

$$\cosh^{-1} z = \ln\left(z + \sqrt{z^2 - 1}\right) \quad \text{for} \quad (1 \leq z < \infty)\,. \tag{10}$$

Therefore,

$$\cosh^{-1} 2 = \ln\left(2 + \sqrt{3}\right). \tag{11}$$

Thus,

$$y = \pm \ln\left(2 + \sqrt{3}\right). \tag{12}$$

Now we use (6) to get x:

$$x = \pm \frac{y}{\ln a} = \begin{cases} +\dfrac{\ln\left(2 + \sqrt{3}\right)}{\ln a} = \dfrac{\ln\left(2 + \sqrt{3}\right)}{\ln\left(\frac{2}{3}\right)} \\ -\dfrac{\ln\left(2 + \sqrt{3}\right)}{\ln a} = \dfrac{\ln\left(2 - \sqrt{3}\right)}{\ln\left(\frac{2}{3}\right)}\,. \end{cases} \tag{13}$$

WolframAlpha declares two solutions:

$$x = \frac{\ln\left(2 - \sqrt{3}\right)}{\ln\left(\frac{3}{2}\right)}\,, \tag{14a}$$

$$x = \frac{\ln\left(2 + \sqrt{3}\right)}{\ln\left(\frac{3}{2}\right)}\,. \tag{14b}$$

We can match one of these solution to (13) by multiplying both numerator and denominator by negative one:

$$x = \frac{-\ln\left(2 + \sqrt{3}\right)}{-\ln\left(\frac{2}{3}\right)} = \frac{\ln\left(2 + \sqrt{3}\right)^{-1}}{\ln\left(\frac{2}{3}\right)^{-1}} = \frac{\ln\left(2 - \sqrt{3}\right)}{\ln\left(\frac{3}{2}\right)}\,. \tag{15}$$

And similarly for the other.

[1]Remember that cosh is an even function of its argument.