Introduction to Mathematical Induction

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Abstract

This paper presents the barest introduction to mathematical induction.

M. Poincaré finds the answer to these questions in the so-called 'mathematical induction' which proceeds from the particular to the more general, but at the same time does so by steps of the highest degree of certitude. In this process he sees the creative force of mathematics, which leads to real proofs and not mere sterile verifications.

— J. W. A. Young

1 Introduction

By the principle of mathematical induction, we can prove certain kinds of claims that could be proved by no other means. This can occur because the claims are a collection of an infinite number of claims, each indexed by an integer, usually successively. For example, consider the claim that

$$1 + 2 + \cdots + n = \frac{n(n+1)}{2}\,, \tag{1}$$

for all integers n starting at 1 and going arbitrarily high. For any particular value of n, this claim is peculiar to that n.

Let's look at this more abstractly. For each integer n, starting at $n = 1$ in this case, the equation in (1) is a proposition in the logical sense. A 'proposition' is a statement (claim) that's either true or false. Let's agree to reference each proposition in (1), indexed by n, as '$P(n)$'. Hence, '$P(1)$' is the proposition that '$1 = \frac{1(1+1)}{2}$' (which is true), '$P(2)$' is the proposition that '$1 + 2 = \frac{2(2+1)}{2}$' (which is true), and so on. The big question is then, how do we show that $P(n)$ is true for all positive integers n?

2 Tailor-made abstract mathematical induction

There are all kinds of situations that can be solved using mathematical induction: the particular form presented in (1) is just one of those forms. And it is to this particular form that I want to develop a standardized means of thinking about it and solving it. But first, some definitions.

I suppose the reader already knows the common acronyms of 'LHS' and 'RHS', which mean, respectively, 'left-hand side' and 'right-hand side', which refer to the two opposing sides of an equality. The function of n on the LHS side of (1) and all propositions like it, will be referred to as $L(n)$; similarly, the function of n on the RHS side of (1) and all propositions like it, will be referred to as $R(n)$. The proposition $P(n)$ that Eq. (1) is true for a given n is equivalent to the claim that

$$L(n) = R(n) \tag{2}$$

for that n.

Now, I'll reveal the steps one must take to apply mathematical induction generally. The 'base case' is the least integer on which the proposition is supposed to apply. In (1) that is $n = 1$. For simplicity, let's refer to the base case as m.

1) Show by brute calculation that $P(m)$ is true.
2) Assume $P(k)$ is true for some arbitrary $k \geq m$.
3) Show, on the basis of 1) and 2), that $P(k+1)$ is true.

Then, if one can successfully comply with all three of these requirements, then one can conclude that $P(n)$ is true for all n greater than or equal to the base case m.

And now, some more definitions. Once more referring to Eq. (1), and all similar problems, we see that the dots on the LHS indicate that a number of terms have been omitted from explicit representation. This is a practical issue to deal with: What if n were a million? $L(n)$ in this case has a special descriptor: it is said to be an 'open form'. That means that without using the ellipses, the function presentation would take on an increasingly large amount of space as n increases. Whereas the form on the RHS is compact, and is said to be a 'closed form'.

So, the real value of the closed form $R(n)$ as opposed to $L(n)$ is its computational efficiency. For large $n >> m$, the number of computational operations to compute $L(n)$ is on the order of n; whereas, for $R(n)$ we need only one addition, one multiplication, and one division.

We can now associate an algebraic interpretation of the above steps. As before, $P(m)$ is true if and only if $L(m) = R(m)$. Assuming that $P(k)$ is true is logically equivalent to assuming that $L(k) = R(k)$. Then, to show that $P(k+1)$ is true we need to show that

$$L(k+1) = R(k+1)\,, \tag{3}$$

given that $L(k) = R(k)$.

The base case in (1) is $n = 1$, and I've already shown that $L(1) = R(1)$. Now to the second step: Assuming $P(k)$ is true for some arbitrary $k \geq 1$ means

$$1 + 2 + \cdots + k = \frac{k(k+1)}{2}\,. \tag{4}$$

I'll now establish $P(k+1)$ by showing that

$$L(k+1) = R(k+1)\,. \tag{5}$$

At this point we should nail down exactly what this looks like, which is[1]

$$1 + 2 + \cdots + k + (k+1) = \frac{(k+1)(k+2)}{2}\,. \tag{6}$$

So, how do we go about doing this? Well, we've already assumed that (4) is true, so let's start there by adding $(k+1)$ to both sides of it, which is a legitimate algebraic maneuver.

$$1 + 2 + \cdots + k + (k+1) = \frac{k(k+1)}{2} + (k+1)\,. \tag{7}$$

In a sense we're halfway finished because the LHS of this is already in the form of $L(k+1)$. So, if we can massage the RHS of (7) to the form $\dfrac{(k+1)(k+2)}{2}$, then we will have successfully completed the proof. Now,

$$\begin{aligned} \frac{k(k+1)}{2} + (k+1) &= \frac{k(k+1)}{2} + \frac{2(k+1)}{2} \\ &= \frac{(k+2)(k+1)}{2} \\ &= \frac{(k+1)(k+2)}{2}\,. \end{aligned} \tag{8}$$

Hence, we've shown that $L(k+1) = R(k+1)$, which was all we had left to do to complete the induction proof.

3 Afterthoughts

If the situation arises in the future, I will explain the difference between 'strong' and 'weak' mathematical induction.

[1] It's important that we don't get confused how we arrived at (6): We did so by replacing k by $k+1$ on both sides of (4).