Basic Matrix Algebra

P. Reany

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Matrix algebra is the language of optimization and machine
learning, enabling us to translate complex problems
into solvable equations and uncover
patterns hidden in data.
—Copilot

Matrix algebra is a huge subject, but I will only present at this time its most basic features, including matrix multiplication and taking the determinant of an $n \times n$ matrix. For purposes of this article, I will represent matrices by capital letters.

So, a **matrix** in an array of rows and columns, whose entries (or components) are typically either real or complex numbers. Let's begin by looking at a generic 2×3 matrix, where the 2 counts the number of rows and the 3 counts the number of columns:

$$A = \begin{pmatrix} a_{11}, a_{12}, a_{13} \\ a_{21}, a_{22}, a_{23} \end{pmatrix} . \tag{1}$$

Next, let's look at a 3×2 matrix B, the point of which is to show how to multiply them together.

$$B = \begin{pmatrix} b_{11}, b_{12} \\ b_{21}, b_{22} \\ b_{31}, b_{32} \end{pmatrix} . \tag{2}$$

Now, it's time to multiply A and B, with A on the left and B on the right.

$$AB = \begin{pmatrix} a_{11}, a_{12}, a_{13} \\ a_{21}, a_{22}, a_{23} \end{pmatrix} \begin{pmatrix} b_{11}, b_{12} \\ b_{21}, b_{22} \\ b_{31}, b_{32} \end{pmatrix} . \tag{3}$$

This example exemplifies the general way that two matrices are to be multiplied together. The matrix on the left must have a same number of columns as the number of rows on the right. It's much easier to grasp how to multiply matrices together by watching a video on it. Here, I'll merely present the end result.

$$AB = \begin{pmatrix} a_{11}b_{11} + a_{12}b_{21} + a_{13}b_{31}, & a_{11}b_{12} + a_{12}b_{22} + a_{13}b_{32} \\ a_{21}b_{11} + a_{22}b_{21} + a_{23}b_{31}, & a_{21}b_{12} + a_{22}b_{22} + a_{23}b_{32} \end{pmatrix} . \tag{4}$$

Let's do a 'dimensional' analysis on this matrix product. We multiplied a 2×3 matrix with a 3×2 matrix and ended up with a 2×2 matrix. In general, if we multiply an $r \times m$ with a $m \times n$, we'll end up with a $r \times n$ matrix.

I've already stated that the set of matrices forms an algebra: What does that mean? The product of any two matrices is another matrix, under the requirement that their product is consistent with the column-row multiplication rule given above. And, assuming this rule is in place, then for matrices A, B, C,

$$(AB)C = A(BC) . \tag{5}$$

In other words, matrix multiplication is associative. However, it is not generally true that multiplication is commutative, thus,

$$AB \neq BA , \tag{6}$$

except in special cases.

Speaking of special cases, in the realm of square matrices, there is a very special matrix, known as the unity (or identity) matrix $\mathbf{I}$, which for a 3×3 matrices looks like this

$$\mathbf{I} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} . \tag{7}$$

So, if M is a 3×3 matrix, then

$$\mathbf{I}M = M\mathbf{I} = M . \tag{8}$$

Similarly, we can define an $n \times n$ identity matrix for any natural number n.

The last thing I want to cover at this time is the *determinant* of a square matrix.

Lemma 1: For M being a 2×2 matrix, its determinant is given by

$$\det(M) = \det \begin{pmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{pmatrix} = m_{11}m_{22} - m_{21}m_{12} . \tag{9}$$

As the dimension of the matrix increases, so too does the complications of performing its determinant.

As we can see, the determinant of a matrix is just the sums and differences of products of the components of the matrix, which themselves are scalars, so the end result is a scalar. That is, a determinant is a scalar quantity.

Lemma 2: Let A and B be any two $n \times n$ matrices. Then

$$\det(AB) = \det(A)\det(B) . \tag{10}$$

Put into words, the determinant of a product of matrices is the product of their determinants.