# The Method of Partial Fractions

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No mystery is closed to an open mind.
— Tim White
TV show *Sightings*

## 1 Introduction

The 'Method of Partial Fractions' is a means of converting a fraction that has a complicated denominator into two or more fractions, each of which have simpler denominators than the original. The technique is purely algebraic. This technique finds use in integration, Laplace transforms, and some other areas of mathematics.

## 2 Getting Started

Let's begin with an example. Our problem is to perform the following integration. (You don't need to know what integration is to follow the presentation.)

$$I = \int \frac{dy}{(1-y)(1+y)} \,. \tag{1}$$

The integrand of this integral is the fraction

$$\frac{1}{(1-y)(1+y)} \,. \tag{2}$$

By using the method of partial fractions we can 'break this fraction up' into two simpler fractions, as follows. We introduce two constants $A, B$, such that

$$\frac{1}{(1-y)(1+y)} = \frac{A}{(1-y)} + \frac{B}{(1+y)} \,, \tag{3}$$

where $A$ and $B$ are real numbers. If we can find a suitable pair $A, B$ that satisfy (3), then we can replace $1/(1-y)(1+y)$ by $A/(1-y) + B/(1+y)$ wherever we want.

To solve for $A$ and $B$, we begin by multiplying through by $(1-y)(1+y)$, leaving us with the polynomial equation in $y$:

$$1 = A(1+y) + B(1-y)\,. \tag{4}$$

To understand the logic of what we're about to do, we need to understand what Eq. (4) is all about. This equation is a polynomial equation in two distinct powers of $y$, namely, $y^1 = y$ and $y^0 = 1$. We solve for $A$ and $B$ by forcing (4) to balance for the amount of $y^1$ and $y^0$ on each side, which will produce for us two equations to solve simultaneously for the two unknowns $A, B$.

Again, both sides of this equation contain both a constant part (a quantity of $y^0 = 1$) and a variable part (a quantity of $y^1 = y$), which will present as two coupled equations to solve algebraically. The following two equations is the result of balancing those two quantities independently across (4):

$$(y^0 \text{ part}) \quad 1 = A + B\,, \tag{5a}$$
$$(y^1 \text{ part}) \quad 0 = A - B\,, \tag{5b}$$

where the first of these equations results from setting the constant parts of the LHS to the RHS, and the second results from setting the first-power in $y$ variable part of the LHS to the variable part of same type of the RHS.

The solution to these two is

$$A = \frac{1}{2} \quad \text{and} \quad B = \frac{1}{2}\,, \tag{6}$$

yielding

$$\frac{1}{(1-y)(1+y)} = \frac{1}{2}\frac{1}{(1-y)} + \frac{1}{2}\frac{1}{(1+y)}\,. \tag{7}$$

I don't know if there is a standard name for the two unknowns $A, B$, but I will refer to them as 'coefficients', which is how I displayed them in the previous equation. And here's the point: To be able to solve for these coefficients, we cannot have either more or less than the number of equations that can be extracted out of the original complicated fraction we start with. Let's introduce another complicated fraction for example.

Say we want to use the method of partial fractions to simplify

$$\frac{1}{(x-2)(x+5)(x-3)}\,. \tag{8}$$

We would begin by writing down

$$\frac{1}{(x-2)(x+5)(x-3)} = \frac{A}{(x-2)} + \frac{B}{(x+5)} + \frac{C}{(x-3)}\,. \tag{9}$$

Next, we would multiply through by $(x-2)(x+5)(x-3)$ (to clear of fractions), to get

$$1 = A(x+5)(x-3) + B(x-2)(x-3) + C(x-2)(x+5)\,, \tag{10}$$

which is, again, a polynomial equation, though this time in the variable $x$. Now, since the highest power of $x$ that appears in this equation (if we multiply it out) is 2, belonging to $x^2$, we expect to derive three equations, corresponding to the three powers of $x$, namely, $x^0, x^1, x^2$. Let's do it.

$$1 = A(x^2 + 2x - 15) + B(x^2 - 5x + 6) + C(x^2 + 3x - 10)\,. \tag{11}$$

That means that we can extract three equations from (10), namely,

$$(x^0 \text{ part}) \quad 1 = -15A + 6B - 10C\,, \tag{12a}$$

$$(x^1 \text{ part}) \quad 0 = 2A - 5B + 3C\,, \tag{12b}$$

$$(x^2 \text{ part}) \quad 0 = A + B + C\,, \tag{12c}$$

Three equations, three unknowns, so we can try to solve this system. Actually, the system has the unique solution

$$A = -\frac{1}{7}\,, \quad B = \frac{1}{56}\,, \quad C = \frac{1}{8}\,, \quad . \tag{13}$$

## 3 Insight from Linear Algebra

From linear algebra, we know that when solving a system of linear equations, we need the same number of unknowns as equations, and vice versa. So, if we have three equations to solve, we need three unknowns to constrain them.

For our third and last example, consider simplifying the fraction

$$\frac{1}{(2x-3)(x+5)^2}\,. \tag{14}$$

Next, let's follow the first example:

$$\frac{1}{(2x-3)(x+5)^2} = \frac{A}{2x-3} + \frac{B}{(x+5)^2}\,. \tag{15}$$

On clearing of fractions, we get

$$1 = A(x+5)^2 + B(2x-3)\,, \tag{16}$$

or

$$2x - 34x^2 - 24x + 9)\,. \tag{17}$$

From this equation we can extract three equations in only two unknowns. That won't do. So, it's back to the drawing board.

What we need is another unknown, so we must add in another term to Eq. (16), but what term? Let's try the following equation.

$$\frac{1}{(2x-3)(x+5)^2} = \frac{A}{2x-3} + \frac{B}{x+5} + \frac{C}{(x+5)^2}\,. \tag{18}$$

Then we clear of fractions.

$$1 = A(x+5)^2 + B(2x-3)(x+5) + C(2x-3)\,. \tag{19}$$

Expanding these products, we have that

$$1 = A(x^2 + 10x + 25) + B(2x^2 + 7x - 15) + C(2x - 3)\,. \tag{20}$$

From this we extract three equations:

$$(x^0 \text{ part}) \quad 1 = 25A - 15B - 3C\,, \tag{21a}$$

$$(x^1 \text{ part}) \quad 0 = 10A + 7B + 2C\,, \tag{21b}$$

$$(x^2 \text{ part}) \quad 0 = A + 2B + 0\,. \tag{21c}$$

WolframAlpha claims that the solutions to this system are

$$A = \frac{4}{169}\,, \quad B = -\frac{2}{169}\,, \quad C = -\frac{1}{13}\,. \tag{22}$$