Pascal's Triangle

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> You have to know what to look for, so you can spot it.
> — Papago Indian drug-enforcement border scout

The purpose of this short and simple essay is to teach the reader how to use Pascal's Triangle to solve for the coefficients of terms in a binomial to a power (where a, b, x, y are real or complex numbers), such as

$$\phi = (x + y)^n \,. \tag{1}$$

The graphical triangle below is due to Blaise Pascal. Let's go through a few rows so we can understand how to use it.[1] So, to begin with, what are the coefficients to the terms in $(a + b)^1$? The exponent of 1 tells us that we need to go to the $n = 1$ row, which tells us that the coefficients for a and b are 1 and 1. Therefore, the coefficient to a is 1 and the coefficient to b is also 1.

Next, what are the coefficients to the terms in $(a + b)^2$? The exponent of 2 tells us that we need to go to the $n = 2$ row, which gives us the coefficients 1 and 2 and 1. To see how this works, let's expand $(a + b)^2$ the old-fashioned way.

$$(a + b)^2 = (a + b)(a + b) = a^2 + ab + ba + b^2 \,. \tag{2}$$

Note: I think it helps to know a few lines of Pascal's Triangle.

$n = 0$: 1

$n = 1$: 1 1

$n = 2$: 1 2 1

$n = 3$: 1 3 3 1

$n = 4$: 1 4 6 4 1

$n = 5$: 1 5 10 10 5 1

$n = 6$: 1 6 15 20 15 6 1

[borrowed from https://www.bedroomlan.org/coding/pascals-triangle-latex]

[1] There's a lot of theory that goes with this triangle, but I won't be addressing it.

But since $ab = ba$, this can be re-expressed as

$$(a+b)^2 = a^2 + 2ab + b^2\,, \tag{3}$$

which has coefficients

$$1 \qquad 2 \qquad 1\,, \tag{4}$$

just like in the triangle.

So, where do the numbers in the triangle come from? Yes, one could just determine the coefficients of, say, $(a+b)^6$ by multiplying the expression out long hand, but there's a simpler method by using the triangle.

Each row of the triangle can be determined easily by the values in the row above it .

So, look at the row $n = 3$ in the triangle. The number 3 is the result of adding together the number 1 immediately above it and to its left to the number 2 immediately above it and to its right. The same pattern works for the second 3 in that row. Likewise, in the next row down,

$$4 = 1+3, \quad 6 = 3+3, \quad 4 = 3+1\,. \tag{5}$$

And the same pattern works for every internal element of every row.

Okay, let's use the triangle to find the correct expansion of $(a+b)^5$:

$$(a+b)^5 = a^5 + 5a^4b + 10a^3b^2 + 10a^2b^3 + 5ab^4 + b^5\,. \tag{6}$$

Lastly, find the coefficient of the term a^4b^2 of the exponentiated binomial $(a+b)^6$.

(Ans: 15.)