Virtual Emplacement, Part 1

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You have to know what to look for so you can spot it.
—Papago Indian drug-enforcement border scout.

Although *virtual emplacement* (VE) is nothing more than using identities, it remains a powerful and surprisingly all-too-easy identity to overlook until you see it done by someone else. The idea behind it is this: If you want something added somewhere in an expression, put it there, but undo the damage done so that the value of the expression is unchanged. What follows is a bunch of examples.

Modulo arithmetic:

If $y \equiv 0 \mod r$ then $x \mod r$ can be replaced by $(x+y) \mod r$ without changing its value.

Algebra:

Perhaps our first introduction to VE occurs in algebra. Two instances stand out right away. They are that you can add 0 to any number or expression and not change its value, and that you can multiply any number or expression by unity and not change its value. Interestingly, these two types of VEs continue to be the dominant forms of VEs throughout mathematics until the introduction of grade-selection in complex and Clifford algebras, which we will get to later.

As an example, consider this familiar problem: Find a single fraction to represent $\frac{1}{3} + \frac{1}{7}$.

$$\frac{1}{3} + \frac{1}{7} = \frac{1}{3} \cdot 1 + \frac{1}{7} \cdot 1 = \frac{1}{3} \cdot \frac{7}{7} + \frac{1}{7} \cdot \frac{3}{3} = \frac{10}{21}\,. \tag{1}$$

We can see in this example another common form of VE, which we'll call the "do-and-undo" VE. Replacing 1 by $\frac{3}{3}$ is multiplying and dividing the number 1 by 3. This form of VE takes its best examples in mixing logs with exponentials, squares with square roots, etc.

As an example, consider the task of writing $\sqrt{x} + \sqrt{y}$, as a dependent on z

and w with $z = x + y$, $w = \sqrt{xy}$, and w, x, y, z positive reals.

$$\sqrt{x} + \sqrt{y} = \sqrt{\left[\sqrt{x} + \sqrt{y}\right]^2} = \sqrt{x + 2\sqrt{xy} + y} = \sqrt{z + 2w}\,. \tag{2}$$

This next technique, called *rationalizing a fraction*, crops up frequently. The basic idea is to clear either the numerator or denominator of radicals. In algebra we learn that writing $1/\sqrt{2}$ is a scorned act. We are told that we *must* rationalize this:

$$\frac{1}{\sqrt{2}} = \frac{1}{\sqrt{2}} \cdot \frac{\sqrt{2}}{\sqrt{2}} = \frac{\sqrt{2}}{2} \tag{3}$$

Great, the world is once again safe for rational thought. It is sufficient to know that these are equivalent expressions.

A little more complicated is the following: For a, b distinct, positive reals

$$\frac{1}{\sqrt{a} + \sqrt{b}} = \frac{1}{\sqrt{a} + \sqrt{b}} \frac{\sqrt{a} - \sqrt{b}}{\sqrt{a} - \sqrt{b}} = \frac{\sqrt{a} - \sqrt{b}}{a - b} \tag{4}$$

Furthermore, we have one of the most prevalent of VEs: For arbitrary reals A, B,

$$|B| = |(B - A) + A| \leq |B - A| + |A|\,. \tag{5}$$

In the so-called "completing the square" we look for a means to replace the expression $x^2 + ax$ by $(x + \lambda)^2$. With $\lambda = a/2$ we have

$$x^2 + ax = x^2 + ax + \tfrac{1}{4}a^2 - \tfrac{1}{4}a^2 = (x + a/2)^2 - \tfrac{1}{4}a^2\,. \tag{6}$$

I'll finish this part with examples from even/odd and periodic functions. If $f(x)$ is even (odd) then $f(x) = f(-x)$ ($f(x) = -f(-x)$). If $f(x)$ is periodic with period τ then $f(x) = f(x + \tau)$.

Plane geometry:

If you use vectors to do plane geometry, you can create a triangle of vectors out of a single vector by a simple VE. Consider the vector in the plane given by $\mathbf{a} - \mathbf{b}$, which is a vector with base at $\mathbf{b}$ and tip at $\mathbf{a}$. Now add to this mix any point $\mathbf{c}$ in the plane not on the line generated by the vector $\mathbf{a} - \mathbf{b}$ and we get a vector equation identifying the sides of triangle $\mathbf{abc}$:

$$\mathbf{a} - \mathbf{b} = (\mathbf{a} - \mathbf{c}) + (\mathbf{c} - \mathbf{b})\,. \tag{7}$$

Trigonometry:

I could spend a lot of time on this section, but I think just a few examples will be representative. We already mentioned even and odd and periodic functions, so let's see some examples:

$$\begin{aligned} &\cos x = \cos(-x)\,, \qquad \sin x = -\sin(-x) \\ &\cos x = \cos(2\pi k + x)\,, \qquad \sin x = \sin(2\pi k + x)\,. \end{aligned} \tag{8}$$

And as an example:

$$\tan x = \frac{\sin x}{\cos x} = \frac{-\sin(x+\pi)}{-\cos(x+\pi)} = \tan(x+\pi)\,. \tag{9}$$

Calculus:

Calculus is loaded with VEs. Let's start with this problem: Show that $\lim_{n\to\infty}\sqrt{n}\big(\sqrt{n+1}-\sqrt{n}\big) = 1/2$.

$$\begin{aligned}
\lim_{n\to\infty}\sqrt{n}\big(\sqrt{n+1}-\sqrt{n}\big) &= \lim_{n\to\infty}\left[\sqrt{n}\big(\sqrt{n+1}-\sqrt{n}\big)\cdot\frac{\sqrt{n+1}+\sqrt{n}}{\sqrt{n+1}+\sqrt{n}}\right] \\
&= \lim_{n\to\infty}\left[\frac{\sqrt{n}(n+1-n)}{\sqrt{n+1}+\sqrt{n}}\right] \\
&= \lim_{n\to\infty}\left\{\left[\frac{\sqrt{n}}{\sqrt{n+1}+\sqrt{n}}\right]\frac{1/\sqrt{n}}{1/\sqrt{n}}\right\} \\
&= \lim_{n\to\infty}\frac{1}{\sqrt{(1+1/n)}+1} = \frac{1}{2}\,. \qquad (10)
\end{aligned}$$

This is a good place to mention that delimiters such as (,) and [,] can be used as VEs, as the brackets were used in this problem.

Now for the chain rule. Let $u = u(t)$ be a differentiable function of t at t_0, and let $z = z(u)$ be a differentiable function of u at $u_0 = u(t_0)$, then $dz/dt = (dz/du)(du/dt)$ where the derivatives are evaluated at $t = t_0$.

$$\begin{aligned}
\frac{dz}{dt} &= \lim_{h\to 0}\frac{z(t_0+h)-z(t_0)}{h} \qquad (11) \\
&= \lim_{h\to 0}\frac{z(u(t_0+h))-z(u(t_0))}{u(t_0+h)-u(t)}\cdot\frac{u(t_0+h)-u(t)}{h} \qquad \text{(note VE)} \qquad (12) \\
&= \lim_{h\to 0}\frac{z(u_0+hu'(t_0))-z(u_0)}{hu'(t_0)}\lim_{h\to 0}\frac{u(t_0+h)-u(t)}{h} = \frac{dz}{du}\cdot\frac{du}{dt}\,, \qquad (13)
\end{aligned}$$

where we used that as $h \to 0$ so does $hu'(t_0)$.

With respect to a standard coordinate system, if the horizontal speed of a baseball is $\dot{x} = 80\,\mathrm{m/s}$ and its vertical speed is $\dot{y} = 17\,\mathrm{m/s}$, what is the angle of elevation of the ball? Solution: The ball is assumed to travel a smooth curve through space, approximately in a vertical plane. Now dy/dx is the instantaneous slope of the tangent line to the curve. Thus,

$$\tan\theta = \frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{\dot{y}}{\dot{x}} = \frac{17}{80}\,. \tag{14}$$

So $\theta = \tan^{-1} 17/80$.

Theorem: If a function $f = f(x)$ is differentiable at a point c, it is continuous at c. Proof: To show that f is continuous at c we must show that

$\lim_{x\to c}[f(x) - f(c)] = 0$, where f is obviously defined at c. Somehow we must use that $\lim_{x\to c}[f(x) - f(c)]/(x - c) = f'(c)$ in our proof. We'll interject it virtually.

$$\lim_{x\to c}[f(x) - f(c)] = \lim_{x\to c}\left[\frac{f(x) - f(c)}{(x - c)} \cdot (x - c)\right] = f'(c) \cdot \lim_{x\to c}(x - c) = 0\,. \quad (15)$$

And we are finished.

Let's look at an example to illustrate this. Say I'm asked to find the limit $\lim_{u\to 0} \frac{\sin u}{\sqrt{u}}$. I say to myself, "Gee, that looks familiar, like $\lim_{u\to 0} \frac{\sin u}{u} = 1$." (Which, if you didn't already know, you do now.) Then I invoke the VE

$$\lim_{u\to 0} \frac{\sin u}{\sqrt{u}} = \lim_{u\to 0} \frac{\sin\ u}{\sqrt{u}} \cdot \frac{\sqrt{u}}{\sqrt{u}} = \lim_{u\to 0} \frac{\sin u}{u} \lim_{u\to 0} \sqrt{u} = 1 \cdot 0 = 0\,. \quad (16)$$

The length of an arc in a plane can be calculated with the aid of VE. Let $y = f(x)$ be a differentiable function defined on an open continuous subset of the x axis. Let ds be an infinitesimal chord having components dx and dy, then the arc length of the curve between any two points a, b on the interval is

$$\int_a^b ds = \int_a^b \sqrt{dx^2 + dy^2} = \int_a^b \frac{\sqrt{dx^2 + dy^2}}{dx}\, dx = \int_a^b \sqrt{1 + (f'(x))^2}\, dx\,. \quad (17)$$

It was when I solved this next problem that I came to fully appreciate the need to formalize the notion of VE. This is definitely a "feel-good" problem. If you want something somewhere, put it there, but undo the damage to maintain equality. Ok, show that

$$\lim_{h\to 0}(1 + h)^{1/h} = e\,. \quad (18)$$

The question is, Where does the e come from?! Who knows, but we can put it there virtually.

$$\lim_{h\to 0}(1 + h)^{1/h} = \lim_{h\to 0} \exp\left\{\ln(1 + h)^{1/h}\right\} = \exp\left\{\lim_{h\to 0} \frac{1}{h} \ln(1 + h)\right\} = e^1 = e\,. \quad (19)$$

Beautiful. (By the way, $\lim\ e^u = \exp\{\lim u\}$ because the exponential function is analytic.)

Alright, y'all warmed up? What is $\int \sec x\, dx$?

$$\int \sec x\, dx = \int \frac{\sec x(\tan x + \sec x)}{\tan x + \sec x}\, dx \quad (20)$$

$$= \int \frac{d(\tan x + \sec x)}{\tan x + \sec x} \quad (21)$$

$$= \ln|\tan x + \sec x| + c\,. \quad (22)$$