Fibonacci Problems #1

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Abstract

There seems to be an endless set of theorems and identities about the Fibonacci numbers. In this paper we will prove a rather simple result about them.

1 Introduction

We will show that successive Fibonacci numbers are relatively prime. I will use induction, but there may be other ways.

2 Getting Started

The Fibonacci sequence is based on the recursive definition: Starting with 0 and 1, add these two numbers to get 1.[1] Add the 1 and the 1 to get 2. Add the 1 and the 2 to get 3. Add the 2 and the 3 to get 5, and we have the start of an infinite sequence of numbers: $\{0, 1, 1, 2, 3, 5, \ldots\}$. The sequence has the simple recursive formula

$$F_{n+2} = F_n + F_{n+1}\,, \tag{1}$$

where $F_0 = 0$, $F_1 = 1$, $F_2 = 0 + 1 = 1$, $F_3 = 1 + 1 = 2$, $F_4 = 1 + 2 = 3$, etc.

Note: The Greatest Common Divisor (GCD) of two numbers a and b is represented as (a, b). If a and b are relatively prime, then $(a, b) = 1$.

Note: To represent that a divides b (evenly) we write $a \,|\, b$.

Lemma: Any-Two-Out-of-Three Lemma:

Let a, b and n be positive integers. Then if n divides any two of a, b, $a + b$, it divides the third.

[1] The literature on Fibonacci numbers shows the starting number as either 0 or 1, but the recurrence definition for these numbers is not affected.

Proof:

Clearly, if n divides both a and b, it divides $a+b$. Let's do the more interesting case. Assume that n divides both a and $a+b$, show that it divides b as well.

Let's convert to a more symbolic representation. We are told that

$$n \,|\, a \quad \text{and} \quad n \,|\, (a+b)\,. \tag{2}$$

But $n \,|\, a$ implies that there exists an integer m such that $a = nm$. And, $n \,|\, (a+b)$ implies that there exists an integer ℓ such that $a+b = n\ell$, hence

$$nm + b = n\ell\,, \tag{3}$$

which can be solved for b:

$$b = n(\ell - m)\,. \tag{4}$$

Hence, $n \,|\, b$, which is what we needed to show.

It's important to take note that if $n \,|\, (a+b)$ it is not necessarily true that $n \,|\, a$ and $n \,|\, b$. For a counterexample, consider that $2 \,|\, 4$, but 4 can be written as $1+3$, and two divides neither of them. That's why it's 'any two out of three'.

3 Proof of the lemma: Successive Fibonacci numbers are relatively prime.

In our induction proof, we begin by showing the base case. We can choose that to be when the number is 2 and its successor is 3: $(2,3) = 1$. Next, we make the so-called inductive hypothesis, by which we assume the result is true for arbitrary case k, which I'll define to be

$$(F_k, F_{k+1}) = 1\,. \tag{5}$$

The goal is to use (5) to show that

$$(F_{k+1}, F_{k+2}) = 1\,. \tag{6}$$

Let's complete the proof by an argument by contradiction. Let's assume that

$$(F_{k+1}, F_{k+2}) = n\,, \tag{7}$$

where $n > 1$. It immediately follows that $n \,|\, F_{k+1}$.

Furthermore, using (1), the recursion relation, we have that

$$(F_{k+1}, F_k + F_{k+1}) = n\,. \tag{8}$$

And we must conclude that $n \,|\, (F_k + F_{k+1})$. From the Any-Two-of-Three Lemma, this implies that $n \,|\, F_k$. And this looks like a problem. We've just shown that $n \,|\, F_k$ and $n \,|\, F_{k+1}$; hence,

$$(F_k, F_{k+1}) = n\,, \tag{9}$$

which contracts our inductive hypothesis. This means that for arbitrary $n \geq 3$

$$(F_n, F_{n+1}) = 1\,. \tag{10}$$

and we are finished.

Other proofs are available at
https://math.stackexchange.com/questions/24378