Problems Concerning the LCM

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Abstract

The LCM refers to the *least common multiple* of a set of positive integers. The LCM of integers a and b is denoted in this paper as $[\,a, b\,]$, while the more familiar relation of GCD is denoted as (a, b). The LCM of two integers is the smallest integer that is evenly divisible by both integers, one at a time.

1 Problem 1

Edwardo wants to prepare a number of gift boxes for the children in an orphanage. He has two rules to follow: 1) He wants all the gift boxes to be identical in content, and 2) he needs to limit the number of gift boxes as a monetary constraint. Now, among the items Edwardo wants in each gift box is a pencil and an eraser. If the pencils are sold by the group, at 20 per package, and the erasers are sold by group, at 12 per package, what is the fewest number of pencil packages and erasure packages Edwardo should purchase to fulfill the rules?

Into each gift box will go one pencil and one eraser, plus other items which we will not track. If Edwardo buys n packages of pencils, that's a total of $20n$ pencils. And, if he buys m packages of erasers, that's a total of $12m$ erasers. To satisfy the constraint of one pencil and one eraser per gift box, we set down the equation

$$20n = 12m = x\,. \tag{1}$$

In other words, the total number of pencils equals the total number of erasers. We can call this number x, just to give it a name.

Whatever the value of x is, it must be divisible by both 20 and by 12, individually. To keep to Rule 2 to conserve money, Edwardo will want the smallest number that 20 and 12 both individually divide, and that is their least common multiple, or the $[\,20, 12\,]$. And this number happens to be 60.

To prove this, let's look at the multiples of 20, which are 20, 40, 60, 80,.... And the multiples of 12 are 12, 24, 36, 48, 60, 72, and so on. And we see now that the smallest entry in both lists is 60.

We're almost finished. To solve for n, we use (1) to get that

$$n = \frac{60}{20} = 3\,. \tag{2}$$

Similarly, to solve for m, we have that

$$m = \frac{60}{12} = 5\,. \tag{3}$$

Thus, Edwardo should buy 3 packages of pencils and 5 packages of erasers.

2 Problem 2

Let m and $a_1, \ldots, a_n$ be positive integers. If m is a multiple of a_i for all $i \in \{1, \ldots, n\}$, then m is a multiple of $[\,a_1, a_2, \ldots, a_n\,]$.

Proof. Stated equivalently, if $a_i \,\big|\, m$ for all $i \in \{1, \ldots, n\}$, then $[\,a_1, a_2, \ldots, a_n\,] \,\big|\, m$. Well, since $[\,a_1, a_2, \ldots, a_n\,] \leq m$, then by the Division Algorithm, there exists some positive integer q, such that

$$m = q \cdot [\,a_1, a_2, \ldots, a_n\,] + r\,, \tag{4}$$

where $0 \leq r < [\,a_1, a_2, \ldots, a_n\,]$.

Now, since by definition, $a_i \,\big|\, [\,a_1, a_2, \ldots, a_n\,]$ for all $i \in \{1, \ldots, n\}$, then we can factor out an a_i on the LHS (for arbitrary i) to get

$$m = a_i \cdot q \cdot \frac{[\,a_1, a_2, \ldots, a_n\,]}{a_i} + r\,, \tag{5}$$

which is true for all i. But we also know that for each i, there exists some α_i such that $m = \alpha_i a_i$. Therefore (5) can be rewritten as

$$\alpha_i a_i = a_i \cdot q \cdot \frac{[\,a_1, a_2, \ldots, a_n\,]}{a_i} + r\,, \tag{6}$$

which, again, is true for all i. On solving this last equation for r, we have that

$$r = a_i \left(\alpha_i - q \cdot \frac{[\,a_1, a_2, \ldots, a_n\,]}{a_i} \right), \tag{7}$$

which implies that a_i divides r for each i. But this means that r is a common multiple of the a_i's. And this means that the least a nonzero r can be is the LCM of the a_i's. But r is supposed to be strictly less than this number. The only way to apply all these conditions consistently is to set $r = 0$.

But, if we set $r = 0$ and put his into (4), we get that

$$m = q \cdot [\,a_1, a_2, \ldots, a_n\,]\,. \tag{8}$$

But this implies that $[\,a_1, a_2, \ldots, a_n\,] \,\big|\, m$ and we are finished.