Problems Concerning the LCM (II)

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Abstract

The LCM refers to the *least common multiple* of a set of positive integers. The LCM of integers a and b is denoted in this paper as $[\,a, b\,]$, while the more familiar relation of GCD (greatest common divisor) is denoted as (a, b). The LCM of two integers is the smallest integer that is evenly divisible by both integers, one at a time.

1 Problem 1

We begin with a simple theorem that relates the LCM of two numbers to their GCD.

Let a and b be two positive integers. Show that

$$[\,a, b\,] = \frac{ab}{(a, b)}\,. \tag{1}$$

Let's begin with the special case when the GCD of a and b is unity, or $(a, b) = 1$. Clearly ab is a common multiple of both a and b, but is it the smallest common multiple of them? Since a and b have no common prime factors, then we need all of the primes in a to divide ab and we need all the prime factors of b to divide the primes in b. Therefore we cannot remove any of the prime factors of ab and still have ab be a common multiple of a and b.

This leads us to conclude that

$$[\,a, b\,] = ab \quad \text{when} \quad (a, b) = 1\,. \tag{2}$$

In our next case, we'll see what will happens when a and b share a single prime factor, which we'll call p. Thus $(a, b) = p$. Then ab will contain two factors of p, one from a and one from b. But we only need one. Therefore

$$[\,a, b\,] = \frac{ab}{p} = \frac{ab}{(a, b)} \quad \text{when} \quad (a, b) = p\,. \tag{3}$$

In our next case, we'll see what will happen when a and b share a single prime factor, which we'll call p, but to possibly different powers. Let the total prime factor of a by p be p^α, and let the total prime factor of b be p^β. And, WLOG, let $\alpha \geq \beta$. Now, we always need the largest power of any given prime in the two numbers, that being p^α, in this case, to be the LCM of any two integers.

So, when we multiply a and b together, we get too many factors of p, that being $p^{\alpha+\beta}$. Just how many extra factors of p is that? Obviously, β factors. So, with respect to the common prime p, we need to divide ab by p^β to remove the over-count. But, quite conveniently, $p^\beta = (a, b)$ in this case.

Therefore

$$[\, a, b\,] = \frac{ab}{p^\beta} = \frac{ab}{(a,b)} \quad \text{when} \quad (a,b) = p^\beta \,. \tag{4}$$

To finish up, consider that a and b have n common prime factors p_i, $i \in [1..n]$, with greatest shared powers β_i. Then we conclude that

$$[\, a, b\,] = \frac{ab}{\prod_i p_i^{\beta_i}} = \frac{ab}{(a,b)} \quad \text{when} \quad (a,b) = \prod_i p_i^{\beta_i} \,. \tag{5}$$

2 Problem 2

Show that for positive integers a_1, a_2, …, a_n, and b:

$$\big[\, [\, a_1, a_2 \ldots, a_n\,], b\,\big] = [\, a_1, a_2 \ldots, a_n, b\,] \,. \tag{6}$$

Proof: We begin by indexing the integers a_i, where $i \in [1..n]$. Now, each a_i can have many prime factors. For my purposes, I'll index the prime factors of $A \equiv a_1 a_2 \cdots a_n$ by j, whereby a prime p_j is in the set of prime factors of A if and only if $p_j \Big| A$, where $j \in [1..m]$ and $m \geq n$.

So, as we saw in the last problem, we need the highest power of every prime that divides the product of all the contributing integer factors. In our problem here, that factor will appear in at least one of the a_i. Let the highest power of p_j that appears in all the a_i's be called α_j. Then that prime contributes the amount $p_j^{\alpha_j}$ to the LCM of all the a_i's, which is its contribution to the number $[\, a_1, a_2 \ldots, a_n\,]$. Therefore

$$[\, a_1, a_2 \ldots, a_n\,] = \prod_j p_j^{\alpha_j} \,. \tag{7}$$

Now, let $b = \prod_k p_k^{\beta_k}$. Therefore

$$\big[\, [\, a_1, a_2 \ldots, a_n\,], b\,\big] = [\, \prod_j p_j^{\alpha_j}, \prod_k p_k^{\beta_k}\,] \,, \tag{8}$$

where I point out that the indexing by j and k are mutually independent. Then

$$[\, \prod_j p_j^{\alpha_j}, \prod_k p_k^{\beta_k}\,] = \prod_\ell p_\ell^{\gamma_\ell} \,, \tag{9}$$

where ℓ runs over all the primes (once) indexed by either j and k, and where γ_ℓ is the old value of either α or β if the prime appears in only one of the lists. But if the prime is in both products, then γ_ℓ is the larger of the two exponents.

I leave it to the reader to decide if I'm right that

$$\prod_\ell p_\ell^{\gamma_\ell} = [\, a_1, a_2 \ldots, a_n, b\,]\,, \tag{10}$$

for then we would conclude that

$$\big[\,[\, a_1, a_2 \ldots, a_n\,], b\big] = [\, a_1, a_2 \ldots, a_n, b\,]\,. \tag{11}$$

3 Problem 3

Let x be a positive number. Solve the following equation for x.

$$[\,20, 15, 12, x\,] = 420\,. \tag{12}$$

From the last problem, we know that

$$[\,20, 15, 12, x\,] = \big[[\,20, 15, 12\,], x\,\big]\,. \tag{13}$$

But

$$[\,20, 15, 12\,] = 60\,. \tag{14}$$

Therefore (12) becomes

$$[\,60, x\,] = \frac{60x}{(60, x)} = 420\,. \tag{15}$$

On simplifying, we get

$$\frac{x}{(60, x)} = 7\,. \tag{16}$$

The obvious answer is $x = 7$.