

Notes for Analytic Number Theory Based on a Video Series by MrYouMath (2012), Lectures on the Riemann-Euler Zeta Function

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Abstract

This paper contains my notes on analytic number theory presented by MrYouMath's video Series (2012). These notes are meant to aid the viewer in following these video presentations, without having to take copious notes. I will likely add to the presented equations and explanations, and some of my own comments and calculations. A knowledge of complex variables is presumed on the part of the reader. The fault for any inaccuracies in these notes is strictly mine.

0 Series Introduction

I intend to motivate this series a little. Since I am not well-trained in number theory, my exposition will not be optimal, but I have looked all over to get a good overview of analytic number theory, but to no avail so far.

Analytic number theory is a subdivision of number theory, whose primary subject of interest are the integers and the functions that can be defined on them. The primary focus of the study of numbers is the properties of the prime numbers. And the most coveted property of the set of prime numbers is its distribution. But since I don't really understand why mathematicians care about the distribution of primes among the natural numbers, I asked ChatGPT for an explanation. On 7 June 2023, my conversation with it went like this:

User

Why do we care about the distribution of prime numbers? What difference does it make?

ChatGPT

The distribution of prime numbers is an intriguing topic in mathematics, and studying it has several implications and practical appli-

cations. Here are a few reasons why we care about the distribution of prime numbers:

Number theory: Prime numbers are fundamental objects in number theory, which is the branch of mathematics that deals with properties and relationships of numbers. Understanding the distribution of primes helps mathematicians explore patterns, develop conjectures, and prove theorems related to prime numbers.

Cryptography: Prime numbers play a crucial role in modern cryptography, which is used to secure communication and protect sensitive information. The security of many cryptographic algorithms relies on the difficulty of factoring large composite numbers into their prime factors. By understanding the distribution of primes, mathematicians and computer scientists can develop more secure encryption methods.

Algorithms and computing: Prime numbers are also relevant in various computational algorithms. For instance, prime numbers are used in generating random numbers, searching large databases efficiently (e.g., prime number sieves), and optimizing algorithms in various fields, including computer science, physics, and engineering.

Mathematics research: Exploring the distribution of prime numbers has led to the discovery of new mathematical concepts and connections between different areas of mathematics. The study of primes has deep connections to other fields, such as algebraic geometry, complex analysis, and representation theory.

Applications in physics: Prime numbers have been found to have unexpected connections to certain physical phenomena, particularly in the field of quantum mechanics. For example, prime numbers appear in the energy levels of certain atomic and molecular systems, leading to insights in quantum chaos and the behavior of quantum systems.

Pure curiosity: Understanding the distribution of prime numbers is a fascinating pursuit in its own right. Humans have an innate curiosity about the patterns and structures that exist in the natural world, and prime numbers offer a rich playground for exploration and discovery.

While the practical applications of prime number distribution may not be immediately apparent in everyday life, they have significant implications for mathematics, computer science, cryptography, and physics. Furthermore, the pursuit of knowledge and understanding, even in seemingly abstract areas, often leads to unexpected breakthroughs and advancements in various fields.

So, according to ChatGPT, these are the ongoing issues and motivations. Although, I must point out that ChatGPT did not stress the distribution of primes

over the other concerns about primes.¹

Mathematicians Euler and Gauss made a lot of noise about what is now call the Euler Zeta function, which is a generalization of the harmonic series. The harmonic series is an infinite sum of fractions in the form

$$\sum_{n=1}^{\infty} \frac{1}{n} = \frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \cdots . \quad (1)$$

This series does not converge. We won't go over the proof of this at this time. Anyway, Gauss showed that the series

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots \quad (2)$$

does converge.

Euler invented a function name for a suitable generalization of the harmonic series, namely the zeta function:

$$\zeta(s) \equiv \sum_{n=1}^{\infty} \frac{1}{n^s} = \frac{1}{1^s} + \frac{1}{2^s} + \frac{1}{3^s} + \cdots , \quad (3)$$

where the parameter s is a real number.

Later on, mathematician Bernhard Riemann generalized the Euler zeta function to his own version by allowing the parameter s to take on complex values, though to force the function to converge, it must have the real part of s to be strictly greater than 1. Thus we have

$$s = x + iy \quad \text{where} \quad x > 1 . \quad (4)$$

So, forcing the real part of s to be greater than 1, gives us a function on the complex plane to the right of the line $x = 1$. Riemann extended this notion of the Euler function to nearly the entire complex plane by a technique called analytic continuation. We may go into this later on.

Throughout his series of video presentation, MrYouMath will gives us a look into some of the most useful function of analytic number theory. Besides the zeta function, he will show us the gamma function, the beta function, and the Jacobi Theta function.

1 Getting Started: Convergence of the Euler Zeta function – Part 1

This series of papers will focus on some of the best known and useful functions of analytic number theory, such as the zeta function, the beta function, the gamma function, and many more. My interest in them originally came from

¹I cannot attest to the accuracy of these points made by ChatGPT.

the fact that these functions have applications in physics. But I have come to find interest in them for the sheer beauty of their theorems, proofs, and wide applicability.

Classically, analytic number theory has its reason for being in the study of the integers, and the study of the integers is centered on the study of primes. And, although we will begin this series with some dazzling results that use the prime numbers, we won't linger on the prime numbers for long.

Some background material on complex numbers is needed. Consider the complex function e^{ix} . Whatever the value of the real number x , the magnitude of e^{ix} is unity, or 1.

$$|e^{ix}| = 1. \quad (5)$$

For a given x value, we can think of e^{ix} as a point of the unit circle in the complex plane.

Another property of complex exponentials we need to know is the simple rule for complex numbers A and B :

$$e^{A+B} = e^A e^B. \quad (6)$$

This simple relation holds true only because the complex numbers is a commutative set. Anyway, if we set $A = x$ and $B = iy$ (where x and y are real numbers), we have that

$$e^{x+iy} = e^x e^{iy}. \quad (7)$$

Next, we need to know how to take the absolute value of a product of complex numbers, which is as follows.

$$|AB| = |A| |B|, \quad (8)$$

for complex numbers A and B . Applying this form to (7) and using (5), we get

$$|e^{x+iy}| = |e^x| |e^{iy}| = |e^x|. \quad (9)$$

So, here's where we begin with the first video of MrYouMath:

We return now the Euler Zeta function:

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \frac{1}{1^s} + \frac{1}{2^s} + \frac{1}{3^s} + \cdots, \quad (10)$$

The method we will use to decide if the Euler Zeta function is convergent, or, more properly, where if anywhere it is convergent, we'll use the well-know notion of absolute convergence. Consider the following series

$$S = \sum_{n=1}^{\infty} s_n. \quad (11)$$

S is said to “converge absolutely” if the following sum converges, i.e., for some positive real number L ,²

$$\sum_{n=1}^{\infty} |s_n| = L. \quad (12)$$

Thus, we write

$$|\zeta(s)| \leq \sum_{n=1}^{\infty} \left| \frac{1}{n^s} \right| = \sum_{n=1}^{\infty} \frac{1}{|n^s|} = \sum_{n=1}^{\infty} \frac{1}{|n^{x+iy}|} = \sum_{n=1}^{\infty} \frac{1}{|n^x| |n^{iy}|}, \quad (13)$$

But, look what we can do now:

$$n^{iy} = e^{iy \ln n}, \quad (14)$$

which is pretty sneaky, if you ask me. Anyway,

$$|n^{iy}| = |e^{iy \ln n}| = 1. \quad (15)$$

Hence, (13) becomes

$$|\zeta(s)| \leq \sum_{n=1}^{\infty} \frac{1}{|n^x|} = \sum_{n=1}^{\infty} \frac{1}{n^x}, \quad (16)$$

which will converge when $x > 1$.

2 Euler Product – Part 2

Well, if you thought that was sneaky, what till you see the so-called ‘Euler Product’. Not willing to be content to write his Euler zeta function as an infinite sum, Euler figured out how to write it equivalently as an infinite product, as follows

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_{p \in P} \frac{1}{1 - p^{-s}}, \quad (17)$$

where P stands for the set of all positive prime numbers. To appreciate the proof, we need to get used to the closed form for what’s called the ‘geometric sum’. So, let x be a complex number and let’s take the sum and get

$$\sum_{n=0}^{\infty} x^n = \frac{1}{1 - x}, \quad (18)$$

and, of course, there is a condition for the convergence of this sum, that being that $|x| < 1$.³

²And we have already started to use the techniques of ‘analysis’ into our discussion.

³The reason this sum is called ‘geometric’ is because the ratio of a term, say, x^{m+1} , to its preceding term, x^m , is a fixed value, in this case being the number x .

To appreciate the RHS of (17), let's start by choosing an arbitrary prime q of the set P and write

$$\frac{1}{1 - q^{-s}}, \quad (19)$$

Now, we fit this into (18) to get

$$\sum_{k=0}^{\infty} (q^{-s})^k = \frac{1}{1 - q^{-s}}, \quad (20)$$

where we have changed the index of summation to k for convenience later on. For the next step, let's index all the positive prime numbers, starting with $i = 1$ and let $q \rightarrow p_i$. Then, we can rewrite the last equation to get

$$\frac{1}{1 - p_i^{-s}} = \sum_{k_i=0}^{\infty} (p_i^{-s})^{k_i} = \sum_{k_i=0}^{\infty} p_i^{-s k_i} = \sum_{k_i=0}^{\infty} \frac{1}{p_i^{s k_i}}, \quad (21)$$

where we have set up a summation index k_i for each prime p_i . Now, we're going back to the RHS of (17):

$$\begin{aligned} \prod_{p \in P} \frac{1}{1 - p^{-s}} &= \prod_{i=1}^{\infty} \frac{1}{1 - p_i^{-s}} = \prod_{i=1}^{\infty} \sum_{k_i=0}^{\infty} \frac{1}{p_i^{s k_i}} \\ &= \sum_{k_1=0}^{\infty} \frac{1}{p_1^{s k_1}} \cdot \sum_{k_2=0}^{\infty} \frac{1}{p_2^{s k_2}} \cdot \sum_{k_3=0}^{\infty} \frac{1}{p_3^{s k_3}} \cdots \\ &= \sum_{k_1=0}^{\infty} \sum_{k_2=0}^{\infty} \sum_{k_3=0}^{\infty} \cdots \left[\frac{1}{p_1^{s k_1}} \cdot \frac{1}{p_2^{s k_2}} \cdot \frac{1}{p_3^{s k_3}} \cdots \right] \\ &= \sum_{k_1=0}^{\infty} \sum_{k_2=0}^{\infty} \sum_{k_3=0}^{\infty} \cdots \left[\frac{1}{p_1^{k_1} p_2^{k_2} p_3^{k_3} \cdots} \right]^s. \end{aligned} \quad (22)$$

Now, here is where the magic happens! Take close look at this infinite product.

$$p_1^{k_1} p_2^{k_2} p_3^{k_3} \cdots, \quad (23)$$

How do we get the number 1 out of this product? Easy. Just set all the k_i 's to zero. To get all the prime numbers out of this, we just set all but one of the k_i 's to zero. The one we don't set to zero, we set to unity. For an arbitrary composite number $N = \prod_{j=1}^m p_{k_j}$, we set to zero every k_i not represented in this product, and set the exponent to all other primes to the value k_j . Therefore, by this procedure, we can construct every natural number exactly once! Therefore, the RHS of (22) becomes

$$\sum_{k_1=0}^{\infty} \sum_{k_2=0}^{\infty} \sum_{k_3=0}^{\infty} \cdots \left[\frac{1}{p_1^{k_1} p_2^{k_2} p_3^{k_3} \cdots} \right]^s = \sum_{n=1}^{\infty} \left[\frac{1}{n} \right]^s = \sum_{n=1}^{\infty} \frac{1}{n^s}. \quad (24)$$

Hence we have that

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_{p \in P} \frac{1}{1 - p^{-s}}. \quad (25)$$

3 Infinitude of primes – Part 4

As a corollary, we know that for $s = 1$, the sum does not converge, therefore the product also cannot converge. However, if there were a finite number of primes, the product

$$\prod_{p \in P} \frac{1}{1 - p^{-s}} \quad (26)$$

must converge! This forces us to conclude that there must be an infinite number of primes. (I hope you can see by now just how sneaky analytic number theory really is. I think Smeagol was on the steering committee.)

4 Euler Product Revisited: Never leave well-enough alone! – Part 3

<https://www.youtube.com/watch?v=TDdGisWD5OU>

So, return we do to the Euler zeta function with the goal to reproduce (25) in another way.

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}. \quad (27)$$

If we multiply this through by $1/2^s$, we get

$$\zeta(s) \frac{1}{2^s} = \sum_{n=1}^{\infty} \frac{1}{n^s} \frac{1}{2^s} = \sum_{n=1}^{\infty} \frac{1}{(2n)^s}. \quad (28)$$

Next, we subtract (28) from (27) to get

$$\zeta(s) \left(1 - \frac{1}{2^s}\right) = \sum_{n=1}^{\infty} \frac{1}{n^s} - \sum_{n=1}^{\infty} \frac{1}{(2n)^s} = \sum_{\substack{n=1 \\ n \neq 2k}}^{\infty} \frac{1}{n^s}, \quad (29)$$

where by $n \neq 2k$ we mean that n may not take on an even value. So, if we employ the same trick for 3 in addition for 2, we get

$$\zeta(s) \left(1 - \frac{1}{2^s}\right) \left(1 - \frac{1}{3^s}\right) = \sum_{\substack{n=1 \\ n \neq 2k \\ n \neq 3k}}^{\infty} \frac{1}{n^s}, \quad (30)$$

If we imagine continuing this process for all primes, the RHS term will boil down to just the first term, where $n = 1$. Therefore,

$$\zeta(s) \left(1 - \frac{1}{2^s}\right) \left(1 - \frac{1}{3^s}\right) \cdots \left(1 - \frac{1}{p_N^s}\right) \cdots = 1, \quad (31)$$

where P_n is the N th prime. And so what's left is an infinite-product form of the Euler Zeta function:

$$\zeta(s) = \prod_{p \in P} \frac{1}{1 - p^{-s}}. \quad (32)$$

5 Prime Zeta Function – Part 5

We need the following result about logarithms

$$\log \frac{1}{1-x} = -\log(1-x) = \sum_{n=1}^{\infty} \frac{x^n}{n} \quad \text{where } |x| < 1. \quad (33)$$

Now, using the property of logarithms that

$$\log AB = \log A + \log B \quad (34)$$

and extending this for an infinite product, we get

$$\begin{aligned} \log \zeta(s) &= \sum_{p \in P} \log \frac{1}{1-p^{-s}} \\ &= \sum_{p \in P} \sum_{n=1}^{\infty} \frac{(p^{-s})^n}{n} \\ &= \sum_{p \in P} \sum_{n=1}^{\infty} \frac{p^{-sn}}{n}. \end{aligned} \quad (35)$$

Next, we can separate out of this summation the $n = 1$ term, leaving us with

$$\log \zeta(s) = \sum_{p \in P} p^{-s} + \sum_{n=2}^{\infty} \frac{1}{n} \sum_{p \in P} \frac{1}{p^{sn}}, \quad (36)$$

where the first term on the RHS is the Euler Zeta function when summed only over the prime numbers and the second term has had the order of summation changed.

Now we have to look at convergence issues. From the second summation of the second term we have that

$$\left| \sum_{p \in P} \frac{1}{p^{sn}} \right| \leq \sum_{p \in P} \frac{1}{|p^{sn}|} = \sum_{p \in P} \frac{1}{|p^{xn+iy n}|} = \sum_{p \in P} \frac{1}{p^{xn}} \leq \sum_{p \in P} \frac{1}{p^n} \leq \sum_{k=2}^{\infty} \frac{1}{k^n}, \quad (37)$$

where we have assumed that $x > 1$.

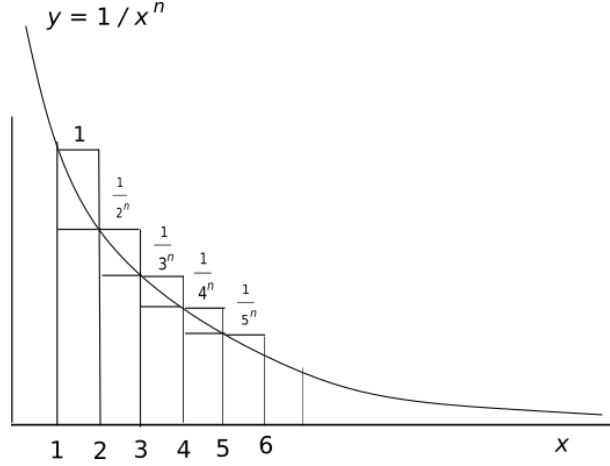


Figure 1. By moving all the rectangles, from the second one on, to the left one unit, we'll be creating a sequence of rectangles whose total area must be less than the area of the curve above it.

So, on returning to (37), we have that

$$\left| \sum_{p \in P} \frac{1}{p^{sn}} \right| \leq \sum_{k=2}^{\infty} \frac{1}{k^n} \leq \sum_{k=2}^{\infty} \int_{k-1}^k \frac{1}{t^n} dt = \int_1^{\infty} \frac{dt}{t^n} = \frac{1}{1-n} t^{1-n} \Big|_1^{\infty} = \frac{1}{n-1}, \quad (38)$$

Therefore,

$$\left| \sum_{n=2}^{\infty} \frac{1}{n} \sum_{p \in P} \frac{1}{p^{sn}} \right| \leq \sum_{n=2}^{\infty} \frac{1}{n} \left| \sum_{p \in P} \frac{1}{p^{sn}} \right| \leq \sum_{n=2}^{\infty} \frac{1}{n} \frac{1}{n-1} = 1, \quad (39)$$

Now, we go back to (36) and take its absolute value, to get

$$|\log \zeta(s)| \leq \left| \sum_{p \in P} p^{-s} \right| + \left| \sum_{n=2}^{\infty} \frac{1}{n} \sum_{p \in P} \frac{1}{p^{sn}} \right|. \quad (40)$$

Now, we know that $\lim_{s \rightarrow 1} \zeta(s) = \infty$ and thus $\lim_{s \rightarrow 1} \log \zeta(s) = \infty$ And thus

$$\left| \sum_{n=2}^{\infty} \frac{1}{n} \sum_{p \in P} \frac{1}{p^{sn}} \right| = 1. \text{ Therefore,}$$

$$\lim_{s \rightarrow 1} \sum_{p \in P} p^{-s} = \sum_{p \in P} \frac{1}{p} \rightarrow \infty. \quad (41)$$

6 The Prime Counting function – Part 6

We now define the Prime Counting function $\pi(n)$ on the positive integers:

$$\pi(n) = \text{the number of primes less than or equal to } n, \quad (42)$$

which is one of the most important functions in all of number theory. Next, we form the difference

$$\pi(n) - \pi(n-1) = \begin{cases} 1 & \text{when } n \text{ is a prime} \\ 0 & \text{otherwise} \end{cases}. \quad (43)$$

We will see that this new difference function is a way to ‘tag’ the primes as distinct from all nonprime numbers.⁴

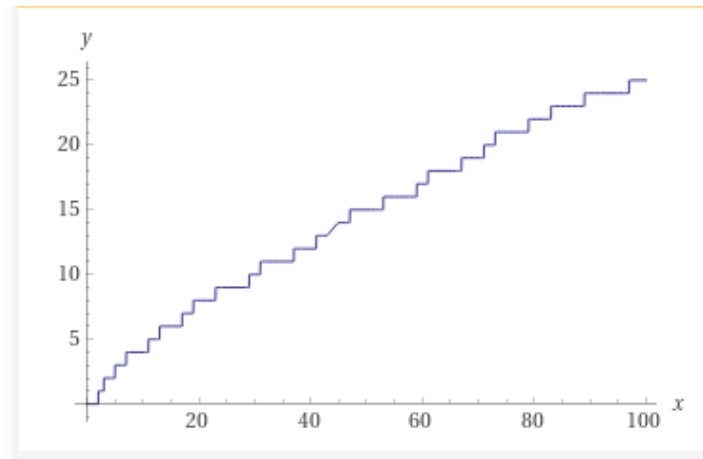


Figure 2. The Prime Counting Function. (Graphic rendered by WolframAlpha.)

We revisit one of our favorite functions and then amend it:

$$\log \zeta(s) = \sum_{p \in P} \log \frac{1}{1-p^{-s}} = \sum_{n=2}^{\infty} (\pi(n) - \pi(n-1)) \log \frac{1}{1-n^{-s}}. \quad (44)$$

So, instead of summing only on the primes we can now sum on all the integers from 2 up. The factor $(\pi(n) - \pi(n-1))$ has the effect of multiplying a prime by unity and a nonprime by zero, effectively removing that term from the summation. (This is one of the great tricks of mathematics.) Anyway, let’s see what

⁴Tagging functions like this are sometimes referred to as ‘indicator functions’ or ‘characteristic functions’.

we can do with this.

$$\begin{aligned}
\log \zeta(s) &= \sum_{n=2}^{\infty} (\pi(n) - \pi(n-1)) \log \frac{1}{1-n^{-s}} \\
&= \sum_{n=2}^{\infty} \pi(n) \log \frac{1}{1-n^{-s}} - \sum_{n=2}^{\infty} \pi(n-1) \log \frac{1}{1-n^{-s}} \\
&= \sum_{n=2}^{\infty} \pi(n) \log \frac{1}{1-n^{-s}} - \sum_{n=2}^{\infty} \pi(n) \log \frac{1}{1-(n+1)^{-s}}
\end{aligned} \tag{45}$$

where, by replacing n by $n+1$ in the second term on the RHS, we lose the first term of the original sum, that being $\pi(1) \log \frac{1}{1-1^{-s}}$; but that term is zero anyway, since $\pi(1) = 0$. So, continuing, we can condense these two summations into just one:

$$\begin{aligned}
\log \zeta(s) &= \sum_{n=2}^{\infty} \pi(n) \log \frac{1}{1-n^{-s}} - \sum_{n=2}^{\infty} \pi(n) \log \frac{1}{1-(n+1)^{-s}} \\
&= \sum_{n=2}^{\infty} \pi(n) \left[\log \frac{1}{1-n^{-s}} - \log \frac{1}{1-(n+1)^{-s}} \right] \\
&= \sum_{n=2}^{\infty} \pi(n) \left[\log(1-(n+1)^{-s}) - \log(1-n^{-s}) \right],
\end{aligned} \tag{46}$$

In the next step, we want to replace this difference of logarithms by a definite integral. Why? Why not? Anyway, consider the following sidebar.

$$D_x \log(1-x^{-s}) = \frac{1}{1-x^{-s}} (sx^{-s-1}) = \frac{s}{x(x^s-1)}. \tag{47}$$

Therefore, on integrating, we get

$$\log(1-x^{-s}) = \int \frac{s}{x(x^s-1)} dx. \tag{48}$$

Now, continuing where we left off and employing the above tricks, we get:

$$\log \zeta(s) = \sum_{n=2}^{\infty} \pi(n) \int_n^{n+1} \frac{s}{x(x^s-1)} dx = \sum_{n=2}^{\infty} \int_n^{n+1} \frac{\pi(n)s}{x(x^s-1)} dx, \tag{49}$$

where we were able to pull the $\pi(n)$ function into the integrand because it is not a function of x . Besides that, we aren't finished with using tricks – oh, no, not by a long shot. There would be no analytic number theory without tricks! We can harmlessly extend the domain of the prime counting function from the positive integers to the positive reals, and get

$$\log \zeta(s) = \sum_{n=2}^{\infty} \int_n^{n+1} \frac{\pi(x)s}{x(x^s-1)} dx. \tag{50}$$

To finish off this section, we need one more trick, but only a small one. The integral of a function from n to $n + 1$ is the area under the curve between the integers n and $n + 1$. But then we add up all these area strips by summing from 2 to ∞ . This means that we can replace the combination of summation and integral with just an integral

$$\log \zeta(s) = \int_2^\infty \frac{s\pi(x)}{x(x^s - 1)} dx. \quad (51)$$

7 $\zeta(2)$ calculated – Part 7

Well, we finally get around to something ‘practical’. But to appreciate this, we need another trick. You remember that the Fundamental Theorem of Algebra is that every polynomial over the complex numbers has a root in the complex numbers. By extension, every polynomial of order n has n roots and thus can be factored into n linear factors.

So, for example, polynomial $f(x) = x^4 + ax^3 + bx^2 + cx + d$ has four roots, say, r_1, r_2, r_3, r_4 . Therefore

$$f(x) = (x - r_1)(x - r_2)(x - r_3)(x - r_4). \quad (52)$$

But why does this work? Simple. By definition, a root r of a function $g(x)$ is a value on which the function becomes zero, or $g(r) = 0$. Okay. Look at (52). If we take $f(r_1)$, we get

$$f(r_1) = (r_1 - r_1)(r_1 - r_2)(r_1 - r_3)(r_1 - r_4) = 0, \quad (53)$$

and likewise for all the other roots.

Okay, so this trick works for any polynomial, but a polynomial has only finite order. That means that a polynomial has a highest power term. Therefore polynomials do not have a sum of progressively higher power to infinity. But that fact is not going to stop us from reasoning analogously

So, we begin with the power series expansion of $\sin x$:

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots, \quad (54)$$

Now we look at a particular argument of this sine function, that being πs and then divide through by πs :

$$\frac{\sin \pi s}{\pi s} = 1 - \frac{(\pi s)^2}{3!} + \frac{(\pi s)^4}{5!} - \frac{(\pi s)^6}{7!} + \dots, \quad (55)$$

The zeros of this function are obviously all the integers except 0, which is ruled out by the denominator. So, let’s just rewrite this function as a product of all its zeros, as if we know what we’re doing. To deal with $s = \pm 1$, we get

$$\frac{\sin \pi s}{\pi s} = \left(1 + \frac{s}{1}\right) \left(1 + \frac{-s}{1}\right) \dots \quad (56)$$

For $s = \pm 2$, we expand this to

$$\frac{\sin \pi s}{\pi s} = \left(1 + \frac{s}{1}\right) \left(1 + \frac{-s}{1}\right) \left(1 + \frac{s}{2}\right) \left(1 + \frac{-s}{2}\right) \cdots. \quad (57)$$

To include $s = \pm 3$, we expand this to

$$\frac{\sin \pi s}{\pi s} = \left(1 + \frac{s}{1}\right) \left(1 + \frac{-s}{1}\right) \left(1 + \frac{s}{2}\right) \left(1 + \frac{-s}{2}\right) \left(1 + \frac{s}{3}\right) \left(1 + \frac{-s}{3}\right) \cdots, \quad (58)$$

and so forth to infinity. But we can simplify this down to

$$\frac{\sin \pi s}{\pi s} = \left(1 - \frac{s^2}{1^2}\right) \left(1 - \frac{s^2}{2^2}\right) \left(1 - \frac{s^2}{3^2}\right) \cdots. \quad (59)$$

Next, we expand the RHS of this in powers of s^2 , to get

$$\frac{\sin \pi s}{\pi s} = 1 - \left(\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots\right) s^2 + \left(\frac{1}{1^2 2^2} + \frac{1}{1^2 3^2} + \frac{1}{2^2 3^2} + \cdots\right) s^4 - \left(\frac{1}{1^2 2^2 3^2} + \cdots\right) s^6 \cdots. \quad (60)$$

Now, if we compare the quadratic term of this last equation with the quadratic term of (55), we get

$$-\frac{\pi^2}{3!} = -\left(\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots\right). \quad (61)$$

And now we are ready to write down $\zeta(2)$

$$\zeta(2) = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots = \frac{\pi^2}{6}. \quad (62)$$

8 $\zeta(2n)$ and the Bernoulli Numbers – Part 8

Let's return to

$$\frac{\sin \pi s}{\pi s} = \prod_{k=1}^{\infty} \left(1 - \frac{s^2}{k^2}\right), \quad (63)$$

and multiply through by πs and then take the logarithm:

$$\log \sin \pi s = \log \pi s + \sum_{k=1}^{\infty} \log \left(1 - \frac{s^2}{k^2}\right). \quad (64)$$

Now we differentiate by s and then multiply through by s :

$$\pi s \cot \pi s = 1 + 2s^2 \sum_{k=1}^{\infty} \frac{1}{s^2 - k^2}. \quad (65)$$

If we take this last equation and divide through by s and reorganize it, we get

$$\pi \cot \pi s = \frac{1}{s} + \sum_{k=1}^{\infty} \frac{2s}{s^2 - k^2} = \frac{1}{s} + \sum_{k=1}^{\infty} \left(\frac{1}{s-k} + \frac{1}{s+k}\right) = \sum_{k=-\infty}^{\infty} \frac{1}{s+k}. \quad (66)$$

Let's return to Eq. (65) and put it into this alternative form:

$$\pi s \cot \pi s = 1 + \sum_{k=1}^{\infty} \frac{1}{(1 - s^2/k^2)} (-2s^2/k^2). \quad (67)$$

Now we replace the factor $\frac{1}{(1 - s^2/k^2)}$ by its equivalence as a geometric series

$$\begin{aligned} \pi s \cot \pi s &= 1 + \sum_{k=1}^{\infty} \sum_{n=0}^{\infty} \left(\frac{s^2}{k^2}\right)^n (-2s^2/k^2) \\ &= 1 - 2 \sum_{k=1}^{\infty} \sum_{n=0}^{\infty} \left(\frac{s^2}{k^2}\right)^{n+1} \\ &= 1 - 2 \sum_{n=1}^{\infty} \left[\sum_{k=1}^{\infty} \frac{1}{k^{2n}} \right] s^{2n} \\ &= 1 - 2 \sum_{n=1}^{\infty} \zeta(2n) s^{2n}. \end{aligned} \quad (68)$$

Therefore, we have the result

$$\pi s \cot \pi s = 1 - 2 \sum_{n=1}^{\infty} \zeta(2n) s^{2n}. \quad (69)$$

Introducing the Bernoulli Numbers

The reason we care about the Bernoulli numbers in this context is because with them we are able to write a formula for the ζ function in terms of them. But to get there, we need some preliminary results.

Remembering that $\cot x = \cos x / \sin x$, we write in terms of their complex counterparts

$$\begin{aligned} \pi s \cot \pi s &= \pi s \frac{e^{i\pi s} + e^{-i\pi s}}{2} \cdot \frac{2}{e^{i\pi s} - e^{-i\pi s}} \\ &= \pi s i \left(1 + \frac{2}{e^{2i\pi s} - 1} \right). \end{aligned} \quad (70)$$

Next, we define the Bernoulli numbers β_n as those numbers that satisfy⁵

$$\frac{s}{e^s - 1} = \sum_{n=0}^{\infty} \frac{\beta_n}{n!} s^n. \quad (71)$$

Moving on. Since

$$e^s = \sum_{n=0}^{\infty} \frac{s^n}{n!} = 1 + \sum_{n=1}^{\infty} \frac{s^n}{n!}, \quad (72)$$

⁵A full treatment of the Bernoulli numbers would answer the question of why the Bernoulli numbers are defined as they are, but that is beyond the scope of this presentation.

then

$$\frac{s}{\sum_{n=1}^{\infty} \frac{s^n}{n!}} = \sum_{m=0}^{\infty} \frac{\beta_m}{m!} s^m, \quad (73)$$

where we changed the index of summation on the RHS so we don't get confused. On rewriting, we get

$$1 = \sum_{m=0}^{\infty} \frac{\beta_m}{m!} s^m \sum_{n=1}^{\infty} \frac{s^{n-1}}{n!}, \quad (74)$$

Next, we employ the Cauchy Product Formula to convert one of the infinite summations to a finite summation.

$$1 = \sum_{n=0}^{\infty} \sum_{\mu=0}^n \frac{\beta_{\mu}}{\mu!} \frac{1}{(n-\mu+1)!} s^n. \quad (75)$$

Now, if we multiply and divide by $(n+1)!$, we have that

$$\begin{aligned} 1 &= \sum_{n=0}^{\infty} \sum_{\mu=0}^n \frac{1}{(n+1)!} \beta_{\mu} \frac{(n+1)!}{\mu!(n-\mu+1)!} s^n \\ &= \sum_{n=0}^{\infty} \sum_{\mu=0}^n \frac{1}{(n+1)!} \beta_{\mu} \binom{n+1}{\mu} s^n \\ &= \sum_{n=0}^{\infty} \frac{1}{(n+1)!} \left[\sum_{\mu=0}^n \binom{n+1}{\mu} \beta_{\mu} \right] s^n. \end{aligned} \quad (76)$$

Obviously, by comparing the LHS to the RHS of this equation the only term on the right that isn't identically zero is the constant term. So, after accounting for the constant term, we get that $\beta_0 = 1$. So, all the higher-order terms in powers of s are zero, therefore

$$\sum_{\mu=0}^n \binom{n+1}{\mu} \beta_{\mu} = 0. \quad (77)$$

Let's work it out some small values of n . For $n = 1$:

$$\binom{2}{0} \beta_0 + \binom{2}{1} \beta_1 = 0. \quad (78)$$

Solving this for β_1 :

$$\beta_1 = -\frac{1}{2} \beta_0 = -\frac{1}{2}. \quad (79)$$

One more. For $n = 2$:

$$\binom{3}{0} \beta_0 + \binom{3}{1} \beta_1 + \binom{3}{2} \beta_2 = 0. \quad (80)$$

Solving this for β_2 :

$$\beta_2 = -\frac{1}{3}[1 + 3(-\frac{1}{2})] = \frac{1}{6}. \quad (81)$$

The odd values of n for $n > 1$ all vanish. And for future needs: $\beta_4 = -1/30$.

Now we return to (70):

$$\pi s \cot \pi s = \pi s i + \frac{2\pi s i}{e^{2i\pi s} - 1}. \quad (82)$$

And, (71) and letting $s \rightarrow \pi s i$, we have that

$$\begin{aligned} \pi s \cot \pi s &= \pi s i + \sum_{n=0}^{\infty} \frac{\beta_n}{n!} (2\pi s i)^n \\ &= \pi s i + \frac{\beta_0}{0!} + \frac{\beta_1}{1!} (2\pi s i) - 2 \sum_{n=2}^{\infty} \left(-\frac{1}{2}\right) \frac{\beta_n}{n!} (2i\pi s)^n \\ &= 1 - 2 \sum_{n=2}^{\infty} \left(-\frac{1}{2}\right) \frac{\beta_n}{n!} (2i\pi s)^n. \end{aligned} \quad (83)$$

Now, since all the β_n 's are zero for odd $n > 1$, then we can continue

$$\begin{aligned} \pi s \cot \pi s &= 1 - 2 \sum_{n=1}^{\infty} \frac{\beta_{2n}}{(2n)!} \left(-\frac{1}{2}\right) (2i\pi s)^{2n} \\ &= 1 - 2 \sum_{n=1}^{\infty} \frac{\beta_{2n}}{(2n)!} (-1) \frac{(2\pi)^{2n} i^{2n}}{2} s^{2n} \\ &= 1 - 2 \sum_{n=1}^{\infty} \frac{(-1)^{n+1} (2\pi)^{2n} \beta_{2n}}{2(2n)!} s^{2n}, \end{aligned} \quad (84)$$

where we used that $i^{2n} = (i^2)^n = (-1)^n$.

Therefore, on comparing this last equation to (69), we have that

$$\zeta(2n) = (-1)^{n+1} \frac{(2\pi)^{2n} \beta_{2n}}{2(2n)!}. \quad (85)$$

And we can test this formula for small values of n : For $n = 1$, we have

$$\zeta(2) = (-1)^1 \frac{(2\pi)^2 \beta_2}{2(2)!} = \frac{\pi^2}{6}. \quad (86)$$

For $n = 2$, we have

$$\zeta(4) = (-1)^3 \frac{(2\pi)^4 \beta_4}{2(4)!} = -\frac{1}{3} \pi^4 \left(-\frac{1}{30}\right) = \frac{\pi^4}{90}. \quad (87)$$

9 The Gamma Function – Part 9

Let s be a complex variable. Then the Gamma function of s is given as

$$\Gamma(s) \equiv \int_0^{\infty} t^{s-1} e^{-t} dt \quad \text{where } \operatorname{Re}(s) > 0. \quad (88)$$

What we're about to do is to show a relationship between the Gamma function and the Zeta function. So, let's begin by making a variable substitution in (88), $t \rightarrow nu$:

$$\begin{aligned} \Gamma(s) &= \int_0^{\infty} (nu)^{s-1} e^{-nu} n du \\ &= \int_0^{\infty} n^s u^{s-1} e^{-nu} du. \end{aligned} \quad (89)$$

Hence,

$$\Gamma(s) \frac{1}{n^s} = \int_0^{\infty} u^{s-1} e^{-nu} du. \quad (90)$$

Does the fraction in the LHS look suggestive in the present context? Either way, let's perform a summation on both sides by n , with it going from 1 to infinity:

$$\Gamma(s) \sum_{n=1}^{\infty} \frac{1}{n^s} = \int_0^{\infty} u^{s-1} \sum_{n=1}^{\infty} e^{-nu} du. \quad (91)$$

By substituting the zeta function on the left and using the geometric series trick on the right, we get

$$\Gamma(s) \zeta(s) = \int_0^{\infty} u^{s-1} \left[\frac{1}{1-e^{-u}} - 1 \right] du = \int_0^{\infty} u^{s-1} \frac{du}{e^u - 1}. \quad (92)$$

10 The Jacobi Theta Function – Part 10

Let's just jump in and define the Jacobi Theta Function

$$\theta(x) \equiv \sum_{n \in \mathbb{Z}} e^{-\pi n^2 x}. \quad (93)$$

But wait! First we need the Poisson Summation Formula:

$$\sum_{n \in \mathbb{Z}} f(n) = \sum_{n \in \mathbb{Z}} \int_{-\infty}^{\infty} f(y) e^{-2\pi i k y} dy. \quad (94)$$

Then,

$$\begin{aligned}
\sum_{n \in \mathbb{Z}} e^{-\pi n^2 x} &= \sum_{k \in \mathbb{Z}} \int_{-\infty}^{\infty} e^{-\pi y^2 x} e^{-2\pi i k y} dy \\
&= \sum_{k \in \mathbb{Z}} \int_{-\infty}^{\infty} e^{-\pi y^2 x - 2\pi i k y} dy \\
&= \sum_{k \in \mathbb{Z}} \int_{-\infty}^{\infty} e^{-\pi \left[(y + ik/x)^2 - i^2 k^2/x^2 \right]} dy \\
&= \sum_{k \in \mathbb{Z}} \int_{-\infty}^{\infty} e^{-\pi k^2/x} e^{-\pi (y + ik/x)^2} dy \\
&= \sum_{k \in \mathbb{Z}} e^{-\pi k^2/x} \int_{-\infty}^{\infty} e^{-\pi (y + ik/x)^2} dy \\
&= \sum_{k \in \mathbb{Z}} e^{-\pi k^2/x} \int_{-\infty + ik/x}^{\infty + ik/x} e^{-\pi x z^2} dz \\
&= \sum_{k \in \mathbb{Z}} e^{-\pi k^2/x} \int_{-\infty}^{\infty} e^{-\pi x z^2} dz. \tag{95}
\end{aligned}$$

The integral on the RHS is known as the Gaussian Integral:

$$\int_{-\infty}^{\infty} e^{-\pi x z^2} dz = \sqrt{\frac{\pi}{\pi x}} = \frac{1}{\sqrt{x}}. \tag{96}$$

So, substituting this into (95) we get

$$\sum_{n \in \mathbb{Z}} e^{-\pi n^2 x} = \sum_{k \in \mathbb{Z}} e^{-\pi k^2/x} \frac{1}{\sqrt{x}}. \tag{97}$$

And so going back to the definition of the theta function, we get

$$\theta(x) = \frac{1}{\sqrt{x}} \theta\left(\frac{1}{x}\right). \tag{98}$$

MrYouMath added these notes:

$$z = y + ik/x \implies dy = dx, \quad |_{-\infty}^{\infty} \rightarrow |_{-\infty + ik/x}^{\infty + ik/x}. \tag{99}$$

$$\sum_{n \in \mathbb{Z}} e^{-\pi n^2 x} = \sum_{k \in \mathbb{Z}} e^{-\pi k^2/x} \int_{-\infty + ik/x}^{\infty + ik/x} e^{-\pi x z^2} dz. \tag{100}$$

And

$$\int_{-R}^R e^{-\pi x z^2} dz = \int_{-R}^{R + ik/x} e^{-\pi x z^2} dz + \int_{-R + ik/x}^{R + ik/x} e^{-\pi x z^2} dz + \int_{R + ik/x}^R e^{-\pi x z^2} dz, \tag{101}$$

where the first and third terms on the RHS are small.

11 The Riemann Functional Equation I (Part 11)

The Riemann Functional Equation looks like the following

$$\pi^{-s/2} \Gamma\left(\frac{s}{2}\right) \zeta(s) = \pi^{-\frac{1-s}{2}} \Gamma\left(\frac{1-s}{2}\right) \zeta(1-s), \quad (102)$$

where we meet the gamma function again:

$$\Gamma(s) = \int_0^{\infty} t^{s-1} e^{-t} dt. \quad (103)$$

Now, if we let $s \rightarrow s/2$:

$$\Gamma\left(\frac{s}{2}\right) = \int_0^{\infty} t^{\frac{s}{2}-1} e^{-t} dt. \quad (104)$$

Next, let $t = \pi n^2 x$:

$$\begin{aligned} \Gamma\left(\frac{s}{2}\right) &= \int_0^{\infty} (\pi n^2 x)^{\frac{s}{2}-1} e^{-\pi n^2 x} \pi n^2 dx \\ &= \int_0^{\infty} \pi^{\frac{s}{2}} n^s x^{\frac{s}{2}-1} e^{-\pi n^2 x} dx. \end{aligned} \quad (105)$$

So, after a little rearranging

$$\pi^{-\frac{s}{2}} \Gamma\left(\frac{s}{2}\right) \frac{1}{n^s} = \int_0^{\infty} x^{\frac{s}{2}-1} e^{-\pi n^2 x} dx. \quad (106)$$

And, if we going to make a connection to the zeta function, we should sum on n :

$$\pi^{-\frac{s}{2}} \Gamma\left(\frac{s}{2}\right) \sum_{n=1}^{\infty} \frac{1}{n^s} = \int_0^{\infty} x^{\frac{s}{2}-1} \sum_{n=1}^{\infty} e^{-\pi n^2 x} dx. \quad (107)$$

Or

$$\pi^{-\frac{s}{2}} \Gamma\left(\frac{s}{2}\right) \zeta(s) = \int_0^{\infty} x^{\frac{s}{2}-1} \sum_{n=1}^{\infty} e^{-\pi n^2 x} dx. \quad (108)$$

And now we bring into this the Jacobi Theta function and the related ψ function:

$$\theta(x) = \sum_{n \in \mathbb{Z}} e^{-\pi n^2 x} = 1 + 2 \sum_{n=1}^{\infty} e^{-\pi n^2 x} = 1 + 2\psi(x), \quad (109)$$

where $\psi(x) \equiv \sum_{n=1}^{\infty} e^{-\pi n^2 x}$. Now we multiply the theta function by $x^{s/2-1}$ and integrate from zero to infinity, then the RHS of (108) takes the form

$$\int_0^{\infty} x^{\frac{s}{2}-1} \psi(x) dx = \int_1^{\infty} x^{\frac{s}{2}-1} \psi(x) dx + \int_0^1 x^{\frac{s}{2}-1} \psi(x) dx, \quad (110)$$

which is true because of the additivity of integrals. Remembering our result that

$$\theta(x) = \frac{1}{\sqrt{x}}\theta\left(\frac{1}{x}\right), \quad (111)$$

and this re-expressed in terms of ψ , yields

$$2\psi(x) + 1 = \frac{1}{\sqrt{x}}(2\psi\left(\frac{1}{x}\right) + 1), \quad (112)$$

or,

$$\psi(x) = \frac{1}{\sqrt{x}}\psi\left(\frac{1}{x}\right) + \frac{1}{2\sqrt{x}} - \frac{1}{2}, \quad (113)$$

Hence, the right-most term of the RHS of (110) becomes

$$\begin{aligned} \int_0^1 x^{\frac{s}{2}-1}\psi(x) dx &= \int_0^1 x^{\frac{s}{2}-1} \left[\frac{1}{\sqrt{x}}\psi\left(\frac{1}{x}\right) + \frac{1}{2\sqrt{x}} - \frac{1}{2} \right] dx, \\ &= \int_0^1 \left[x^{\frac{s}{2}-\frac{3}{2}}\psi\left(\frac{1}{x}\right) + \frac{1}{2} \left(x^{\frac{s}{2}-\frac{3}{2}} - x^{\frac{s}{2}-1} \right) \right] dx \\ &= \int_0^1 x^{\frac{s}{2}-\frac{3}{2}}\psi\left(\frac{1}{x}\right) dx + \frac{1}{2} \left[\frac{1}{s/2-1/2} x^{\frac{s}{2}-\frac{1}{2}} - \frac{1}{s/2} x^{\frac{s}{2}} \right]_0^1 \\ &= \int_0^1 x^{\frac{s}{2}-\frac{3}{2}}\psi\left(\frac{1}{x}\right) dx + \frac{1}{s(s-1)}, \end{aligned} \quad (114)$$

Now we perform a variable transformation of $x = \frac{1}{u}$, $dx = -\frac{1}{u^2} du$ with $|_0^1 \rightarrow |_\infty^1$:

$$\int_0^1 x^{\frac{s}{2}-\frac{3}{2}}\psi\left(\frac{1}{x}\right) dx + \frac{1}{s(s-1)} \rightarrow \int_\infty^1 \left(\frac{1}{u}\right)^{\frac{s}{2}-\frac{3}{2}}\psi(u) \frac{-du}{u^2} + \frac{1}{s(s-1)}, \quad (115)$$

And now let $u \rightarrow x$ and use the minus sign to flip the limits of integration brings (114) to:

$$\int_0^1 x^{\frac{s}{2}-1}\psi(x) dx = \int_1^\infty x^{-\frac{s}{2}-\frac{1}{2}}\psi(x) dx + \frac{1}{s(s-1)}. \quad (116)$$

Substituting this into (110), we have that

$$\begin{aligned} \int_0^\infty x^{\frac{s}{2}-1}\psi(x) dx &= \int_1^\infty x^{\frac{s}{2}-1}\psi(x) dx + \int_1^\infty x^{-\frac{s}{2}-\frac{1}{2}}\psi(x) dx + \frac{1}{s(s-1)} \\ &= \int_1^\infty \left[x^{\frac{s}{2}-1} + x^{-\frac{s}{2}-\frac{1}{2}} \right] \psi(x) dx + \frac{1}{s(s-1)}, \end{aligned} \quad (117)$$

Remember that

$$\pi^{-s/2}\Gamma\left(\frac{s}{2}\right)\zeta(s) = \int_0^\infty x^{\frac{s}{2}-1}\psi(x) dx. \quad (118)$$

Therefore,

$$\pi^{-s/2} \Gamma\left(\frac{s}{2}\right) \zeta(s) = \int_1^\infty [x^{\frac{s}{2}-1} + x^{\frac{1-s}{2}}] \frac{\psi(x)}{x} dx + \frac{1}{s(s-1)}. \quad (119)$$

Next, notice that the entire RHS is invariant if we let $s \rightarrow s-1$, then

$$\pi^{-s/2} \Gamma\left(\frac{s}{2}\right) \zeta(s) = \pi^{-\frac{1-s}{2}} \Gamma\left(\frac{1-s}{2}\right) \zeta(1-s). \quad (120)$$

12 The Zeta Function: Riemann Functional Eq. 2 (Part 12)

We will use the Riemann Functional Eq. (120) to derive its alternative form:

$$\zeta(s) = 2^s \pi^{s-1} \sin \frac{\pi s}{2} \Gamma(1-s) \zeta(1-s). \quad (121)$$

To accomplish this, we need two other results. To get the first one, we begin with the Legendre Duplication Formula,⁶

$$\Gamma(2s) = \frac{2^{2s-1}}{\sqrt{\pi}} \Gamma(s) \Gamma(s+1/2), \quad (122)$$

and then replace s by $s/2$ and rearrange factors to get:

$$\frac{\sqrt{\pi}}{2^{s-1}} \Gamma(s) = \Gamma\left(\frac{s}{2}\right) \Gamma\left(\frac{s+1}{2}\right). \quad (123)$$

Next, we need to get:

$$\Gamma\left(\frac{s+1}{2}\right) \Gamma\left(\frac{s}{2}\right) = \frac{\pi}{\cos \frac{\pi s}{2}}. \quad (124)$$

We'll accomplish this by starting with the Euler reflection formula

$$\Gamma(s) \Gamma(1-s) = \frac{\pi}{\sin \frac{\pi s}{2}}, \quad (125)$$

and then let $s \rightarrow (s+1)/2$:

$$\Gamma\left(\frac{s+1}{2}\right) \Gamma\left(1 - \frac{s+1}{2}\right) = \frac{\pi}{\sin\left(\frac{\pi s}{2} + \frac{\pi}{2}\right)}, \quad (126)$$

⁶See MrYouMath: Gamma Function - Part 11 - Legendre Duplication Formula, Don't get confused! MrYouMath has two related series: one on the Riemann Zeta function and another one on the Gamma Function.

which simplifies to

$$\Gamma\left(\frac{s+1}{2}\right)\Gamma\left(\frac{1-s}{2}\right) = \frac{\pi}{\cos\frac{\pi s}{2}}. \quad (127)$$

Now, multiply (120) through by $\Gamma\left(\frac{s+1}{2}\right)$:

$$\pi^{-s/2}\Gamma\left(\frac{s}{2}\right)\Gamma\left(\frac{s+1}{2}\right)\zeta(s) = \pi^{-\frac{1-s}{2}}\Gamma\left(\frac{s+1}{2}\right)\Gamma\left(\frac{1-s}{2}\right)\zeta(1-s). \quad (128)$$

So now we can use the Legendre duplication formula on the LHS and on the RHS we'll use (127):

$$\pi^{-s/2}\frac{\sqrt{\pi}}{2^{s-1}}\Gamma(s)\zeta(s) = \pi^{-\frac{1-s}{2}}\frac{\pi}{\cos\frac{\pi s}{2}}\zeta(1-s). \quad (129)$$

After some algebra, we have that

$$\zeta(1-s) = \frac{2}{(2\pi)^s}\cos\frac{\pi s}{2}\Gamma(s)\zeta(s). \quad (130)$$

Next, we do the replacement: $1-s \rightarrow s$. Then,

$$\zeta(s) = \frac{2}{(2\pi)^{1-s}}\cos\frac{\pi(1-s)}{2}\Gamma(1-s)\zeta(1-s). \quad (131)$$

On using standard trig identities as before, we have that

$$\zeta(s) = 2^s\pi^{s-1}\sin\frac{\pi s}{2}\Gamma(1-s)\zeta(1-s), \quad (132)$$

which concludes the establishment of (121).

13 Trivial Zeros of the Zeta Function (Part 13)

It's obvious from (132) that because the sine function has zeros at integer multiples of π , that the zeta function has zeros for $s = -2k$ where $k \in \mathbb{Z}^+$. Or,

$$\zeta(-2k) = 0. \quad (133)$$

These zeroes are referred to as 'trivial' because it requires almost no effort to establish this fact.

14 Zeta Function – Riemann Xi Function (Part 14) Splitting the Difference

First, we return to the Riemann Functional Equation:

$$\pi^{-s/2}\Gamma\left(\frac{s}{2}\right)\zeta(s) = \int_1^\infty \left[x^{\frac{s}{2}} + x^{\frac{s-1}{2}}\right] \frac{\psi(x)}{x} dx - \frac{1}{s(1-s)}, \quad (134)$$

Now, multiply through by $\frac{1}{2}s(1-s)$:

$$\frac{1}{2}s(s-1)\pi^{-s/2}\Gamma\left(\frac{s}{2}\right)\zeta(s) = \frac{1}{2}s(s-1) \int_1^\infty \left[x^{\frac{s}{2}} + x^{\frac{s-1}{2}}\right] \frac{\psi(x)}{x} dx + \frac{1}{2}, \quad (135)$$

Define $\xi(s)$ by

$$\xi(s) \equiv \frac{1}{2}s(s-1)\pi^{-s/2}\Gamma\left(\frac{s}{2}\right)\zeta(s), \quad (136)$$

From this we get that

$$\xi(s) = \xi(1-s), \quad (137)$$

Now, let

$$s = \frac{1}{2} + it, \quad \text{where } t \in \mathbb{C}. \quad (138)$$

Then

$$\xi\left(\frac{1}{2} + it\right) = \xi\left(\frac{1}{2} - it\right), \quad (139)$$

15 Afterword

I love watching videos of technical subjects, like math and science, when they are presented well. I have made and published a number of read-along notes to math and physics video series, and I'm glad to add this one by MrYouMath.

By the way, MrYouMath also did a series on the Gamma Function, which I hope to publish soon, as well, in similar read-along format.