Notes for Analytic Number Theory Based on a Video Series by MrYouMath (2012), Lectures on the Gamma Function

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Abstract

This paper contains my notes on analytic number theory presented by MrYouMath's video Series (2012). These notes are meant to be read-along notes to aid the viewer in following these video presentations, without having to take copious notes. I will likely add to the presented equations and explanations, and some of my own comments and calculations. A knowledge of complex variables is presumed on the part of the reader. The fault for any inaccuracies in these notes is strictly mine.

1 The Gamma Function – Part 1

Let s be a complex variable. Then the Gamma function of s is given as

$$\Gamma(s) \equiv \int_0^\infty t^{s-1} e^{-t} dt \quad \text{where } \mathrm{Re}(s) > 0\,. \tag{1}$$

Something profound is the result of merely letting $s \to s+1$ and then integrating by parts:

$$\begin{aligned} \Gamma(s+1) &= \int_0^\infty t^s e^{-t} dt \\ &= -e^{-t} t^s \big|_0^\infty - \int_0^\infty \frac{dt^s}{dt} \left(-e^{-t}\right) dt \\ &= \int_0^\infty s t^{s-1} e^{-t} dt \\ &= s \int_0^\infty t^{s-1} e^{-t} dt \\ &= s\Gamma(s)\,. \end{aligned} \tag{2}$$

Hence, for n a positive integer:

$$\Gamma(n+1) = n!\,. \tag{3}$$

2 Gauss Representation – Part 2

We first need a lemma:

$$e^{-t} = \lim_{n\to\infty}\left(1+\frac{-t}{n}\right)^{n} . \tag{4}$$

Then

$$\begin{aligned}\Gamma(s) &= \int_0^\infty t^{s-1}\lim_{n\to\infty}\left(1-\frac{t}{n}\right)^{n} dt\\ &= \lim_{n\to\infty}\int_0^\infty t^{s-1}\left(1-\frac{t}{n}\right)^{n} dt\,. \end{aligned}\tag{5}$$

Let's see what happens if we try an integration by parts (repeatedly).

$$\begin{aligned}\int_0^n t^{s-1}\left(1+\frac{-t}{n}\right)^{n} dt &= \cancelto{0}{\frac{1}{s}t^s\left(1-\frac{t}{n}\right)^{n}\Bigg|_{t=0}^{n}} + \int_0^n \frac{1}{s}t^s\left(1+\frac{-t}{n}\right)^{n-1}\left(-\frac{1}{n}\right) dt\\ &= \frac{n}{sn}\int_0^n \frac{1}{s}t^s\left(1+\frac{-t}{n}\right)^{n-1}\left(-\frac{1}{n}\right) dt\\ &= \frac{n}{sn}\frac{n-1}{(s+1)n}\int_0^n \frac{1}{s}t^{s+1}\left(1-\frac{t}{n}\right)^{n-2} dt\\ &= \cdots\\ &= \frac{n}{sn}\frac{n-1}{(s+1)n}\frac{n-2}{(s+2)n}\cdots\frac{1}{(s+n-1)n}\int_0^n \frac{1}{s}t^{s+n-1}dt\\ &= \frac{n}{sn}\frac{n-1}{(s+1)n}\frac{n-2}{(s+2)n}\cdots\frac{1}{(s+n-1)n}\frac{t^{s+n}}{s+n}\Bigg|_0^n . \end{aligned}\tag{6}$$

Or,

$$\int_0^n t^{s-1}\left(1+\frac{-t}{n}\right)^{n} dt = \frac{n!}{n^n}n^{s+n}\prod_{i=0}^{n}(s+i)^{-1} = \frac{n!n^s}{\displaystyle\prod_{i=0}^{n}(s+i)} . \tag{7}$$

Therefore, after separating out the $i=0$ factor of the product, we have that

$$\Gamma(s) = \lim_{n\to\infty}\left[\frac{n^s}{s}\prod_{i=1}^{n}\frac{i}{(s+i)}\right], \tag{8}$$

and the $n!$ was replaced by i in the numerator of the product.

3 Weierstrass Representation – Part 3

The Weierstrass Representation of the Gamma function is

$$\Gamma(s) = \frac{1}{s} e^{-\gamma s} \prod_{i=1}^{\infty} e^{\frac{s}{i}} \left(1 + \frac{s}{i}\right)^{-1} , \tag{9}$$

where γ is the Euler-Macheroni constant, defined by

$$\gamma \equiv \lim_{n\to\infty} \left(\sum_{i=1}^{n} \frac{1}{i} - \log n \right) , \tag{10}$$

To prove this, we begin with the Gauss representation and massage it. Note that $e^0 = 1$.[1]

$$\begin{aligned} \Gamma(s) &= \lim_{n\to\infty} \left[\frac{n^s}{s} \prod_{i=1}^{n} \frac{i}{(s+i)} \right] \\ &= \lim_{n\to\infty} \left[\frac{n^s}{s} \prod_{i=1}^{n} \left(1 + \frac{s}{i}\right)^{-1} \right] \\ &= \lim_{n\to\infty} \left[\frac{1}{s} e^0 e^{s \log n} \prod_{i=1}^{n} \left(1 + \frac{s}{i}\right)^{-1} \right] \\ &= \lim_{n\to\infty} \left[\frac{1}{s} e^{\left(\sum_{i=1}^{n} \frac{s}{i} - \sum_{i=1}^{n} \frac{s}{i} \right) + s \log n} \prod_{i=1}^{n} \left(1 + \frac{s}{i}\right)^{-1} \right] \\ &. \end{aligned} \tag{11}$$

Or,

$$\begin{aligned} \Gamma(s) &= \lim_{n\to\infty} \left[\frac{1}{s} e^{\left(\sum_{i=1}^{n} \frac{s}{i} - \gamma s \right)} \prod_{i=1}^{n} \left(1 + \frac{s}{i}\right)^{-1} \right] \\ &= \lim_{n\to\infty} \left[\frac{1}{s} e^{-\gamma s} \prod_{i=1}^{n} e^{s/i} \left(1 + \frac{s}{i}\right)^{-1} \right] \\ &, \end{aligned} \tag{12}$$

where we used that, for the sum on i:

$$e^{\sum \frac{s}{i}} = \prod e^{\frac{s}{i}} . \tag{13}$$

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[1] We will use the trick that $n^s = e^{s \log n}$.

4 Gamma and sine functions together – Part 4

Starting with

$$\Gamma(s)=\lim_{n\to\infty}\left[\frac{n^s}{s}\prod_{i=1}^{n}\frac{i}{(s+i)}\right]=\lim_{n\to\infty}\left[\frac{n^s}{s}\prod_{i=1}^{n}\frac{1}{(1+s/i)}\right], \tag{14}$$

and replacing s by its negative, we have that

$$\Gamma(-s)=\lim_{n\to\infty}\left[\frac{n^{-s}}{-s}\prod_{i=1}^{n}\frac{1}{(1-s/i)}\right]. \tag{15}$$

On multiplying these together and rearranging, we have that

$$\Gamma(s)\Gamma(-s)(-s)=\lim_{n\to\infty}\left[\frac{1}{s}\prod_{i=1}^{n}\frac{1}{(1-s^2/i^2)}\right]. \tag{16}$$

This gives us

$$\Gamma(s)\Gamma(1-s)=\frac{1}{s}\prod_{i=1}^{n}\frac{1}{(1-s^2/i^2)}\,. \tag{17}$$

And, even simpler,[2]

$$\Gamma(s)\Gamma(1-s)=\frac{\pi}{\sin\pi s}\,. \tag{18}$$

5 Gamma of 1/2 – Part 5

Referencing the last equation (18) and substituting into it $s=\frac{1}{2}$, we get

$$\left[\Gamma(\tfrac{1}{2})\right]^2=\pi\,, \tag{19}$$

from we we get the simpler expression

$$\Gamma(\tfrac{1}{2})=\sqrt{\pi}\,. \tag{20}$$

Okay, let's leverage what we've got:

$$\Gamma(\tfrac{1}{2})=\int_0^\infty t^{\frac{1}{2}-1}e^t\,dt=\int_0^\infty \frac{1}{\sqrt{t}}e^t\,dt\,. \tag{21}$$

Therefore,

$$\int_0^\infty \frac{1}{\sqrt{t}}e^{-t}\,dt=\sqrt{\pi}\,. \tag{22}$$

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[2] See Lecture 7 of the Riemann Zeta Function, or refer to the Read-Along notes.

So, what else can we get out of this? Let $t \to pu^2$ with $p > 0$:

$$2p \int_0^\infty e^{-pu^2}\, dt = \sqrt{\pi}\,. \tag{23}$$

From this we get the more symmetric integral

$$\int_{-\infty}^\infty e^{-pu^2}\, dt = \sqrt{\pi/p}\,. \tag{24}$$

This is the so-called Gaussian Integral.

6 The Stirling Approximation – Part 6

The following is the The Stirling Approximation to $n!$:

$$n! \approx \frac{n^n}{e^n}\sqrt{2n\pi}\,. \tag{25}$$

We will build up this result through a long process. We begin with

$$\log\left(t^n e^{-t}\right) = n \log t - t\,. \tag{26}$$

Now, let $t \to n + \epsilon$:

$$n \log t - t = n \log\left(n + \epsilon\right) - \left(n + \epsilon\right). \tag{27}$$

But

$$\log\left(n + \epsilon\right) = \log n(1 + \epsilon/n) = \log n + \log\left(1 + \epsilon/n\right). \tag{28}$$

Now, for $n >> 1$, ϵ/n is small, hence

$$\log\left(1 + \epsilon/n\right) = \sum_{k=1}^\infty \frac{(-1)^{k+1}}{k}\frac{\epsilon^k}{n^k} \tag{29}$$

can be approximated by just a few terms. So

$$\begin{aligned} n \log\left(n + \epsilon\right) - \left(n + \epsilon\right) &= n\left(\log n + \sum_{k=1}^\infty \frac{(-1)^{k+1}}{k}\frac{\epsilon^k}{n^k}\right) - n - \epsilon \\ &= n \log n - n + \sum_{k=1}^\infty \frac{(-1)^{k+1}}{k}\frac{\epsilon^k}{n^{k-1}} - \epsilon \\ &= n \log n - n - \frac{\epsilon^2}{2n} + \frac{\epsilon^3}{3n^2} - \frac{\epsilon^4}{4n^3} + \cdots\,. \end{aligned} \tag{30}$$

Also

$$\log\left(t^n e^{-t}\right) \approx n \log n - n - \frac{\epsilon^2}{2n}\,. \tag{31}$$

Therefore,

$$t^n e^{-t} \approx \frac{n^n}{e^n}\, e^{-\epsilon^2/2n}\,. \tag{32}$$

Thus,

$$n! = \int_0^\infty t^n\, e^{-t}\, dt \approx \int_{-n}^\infty \frac{n^n}{e^n}\, e^{-\epsilon^2/2n}\, d\epsilon\,. \tag{33}$$

And we see the Gaussian integral in variable ϵ. So,

$$n! \approx \frac{n^n}{e^n}\sqrt{2n\pi}\,. \tag{34}$$

7 The Euler Integral I – Part 7

Once again we begin with

$$\Gamma(s) = \int_0^\infty t^{s-1} e^{-t} dt\,, \tag{35}$$

and make the substitution $t \to pu^n$ where p is complex. Then

$$\Gamma(s) = \int_0^\infty np^s u^{ns-1}\, e^{-pu^n}\, du\,, \tag{36}$$

which can be put into the alternative form

$$\frac{\Gamma(s)}{np^s} = \int_0^\infty u^{ns-1}\, e^{-pu^n}\, du\,. \tag{37}$$

Okay, we will denote the complex conjugate by an overbar. In the last equation we can replace p by its complex conjugate, to get[3]

$$\frac{\Gamma(s)}{n\overline{p}^s} = \int_0^\infty u^{ns-1}\, e^{-\overline{p}u^n}\, du\,. \tag{38}$$

On taking a combination of the last two equations, we get

$$\frac{\Gamma(s)}{np^s} \pm \frac{\Gamma(s)}{n\overline{p}^s} = \int_0^\infty u^{ns-1}\, e^{-pu^n}\, du \pm \int_0^\infty u^{ns-1}\, e^{-\overline{p}u^n}\, du\,. \tag{39}$$

Now, if we write p in the polar form, we get

$$p = |\,p\,|\, e^{i\alpha} = a + ib\,. \tag{40}$$

With this result, (39) becomes

$$\frac{\Gamma(s)}{n\,|\,p\,|^s}\left[\frac{1}{e^{-ias}} \pm \frac{1}{e^{i\alpha s}}\right] = \int_0^\infty u^{ns-1}\, e^{-\alpha u^n}\left(e^{isu^n} \pm e^{-isu^n}\right) du\,. \tag{41}$$

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[3] We can do this because p was an arbitrary complex number to begin with.

By taking first the + sign and then the − sign, and stacking them:

$$\frac{\Gamma(s)}{n\,|\,p\,|^s}\begin{bmatrix}\cos\,(\alpha s)\\ \sin\,(\alpha s)\end{bmatrix} = \int_0^\infty u^{ns-1}e^{-au^n}\begin{bmatrix}\cos\,(bu^n)\\ \sin\,(bu^n)\end{bmatrix}du\,, \tag{42}$$

where

$$\begin{cases}\tan\alpha = b/a & a\neq 0\,,\\ \alpha = \pi/2 & a = 0\,.\end{cases} \tag{43}$$

8 The Euler Integral II (the Sinc function) – Part 8

The sinc function is given as $\operatorname{sinc} x = (\sin x)/x$. Its integral is

$$\int_0^\infty \frac{\sin x}{x}\,dx = \frac{\pi}{2}\,. \tag{44}$$

Using the final equation of the last section, and setting

$$a = 0\,,\quad b = 1\,,\quad n = 1\,,\quad s = 0\quad \text{(with effect taken in the limit)}\,, \tag{45}$$

and noting that $|\,p\,| = \sqrt{a^2+b^2} = 1$, and $\alpha = \pi/2$, then from

$$\Gamma(s)\Gamma(1-s) = \frac{\pi}{\sin\pi s}\,, \tag{46}$$

we get

$$\sin(\pi s/2)\cdot\frac{\pi}{\sin\pi s\,\Gamma(1-s)} = \int_0^\infty (u^{s-1}\sin u)\,du\,. \tag{47}$$

With slight modification, we get

$$\frac{\pi}{2}\left[\frac{\sin\,(\pi s/2)}{\pi s/2}\,\frac{\pi s}{\sin\pi s\,\Gamma(1-s)}\right] = \int_0^\infty (u^{s-1}\sin u)\,du\,. \tag{48}$$

Next, we use that $\lim\limits_{x\to 0}\dfrac{\sin x}{x} = 1$, so

$$\lim_{s\to 0}\left[\frac{\sin\,(\pi s/2)}{\pi s/2}\,\frac{\pi s}{\sin\pi s\,\Gamma(1-s)}\right] = \frac{1}{\Gamma(1)} = 1\,. \tag{49}$$

Then,

$$\int_0^\infty \frac{\sin u}{u}\,du = \frac{\pi}{2}\,. \tag{50}$$

9 The Euler Integral III (Fresnel Integrals) – Part 9

The Fresnel Integrals have the forms

$$\begin{aligned}\int_0^\infty \sin u^2\, du &= \frac{\sqrt{2\pi}}{4}\,,\\ \int_0^\infty \cos u^2\, du &= \frac{\sqrt{2\pi}}{4}\,. \end{aligned}\tag{51}$$

Let's bring out (42)

$$\frac{\Gamma(s)}{n\,|\,p\,|^s}\sin\,(\alpha s) = \int_0^\infty u^{ns-1}e^{i\alpha u^n}\sin\,(bu^n)\,du\,. \tag{52}$$

But we have need of specific values:

$$s=\tfrac{1}{2}\,,\quad \Gamma(\tfrac{1}{2})=\sqrt{\pi}\,,\quad a=0\,,\quad b=1\,,n=2\,,\quad \alpha=\pi/2\,. \tag{53}$$

On substituting in, we get

$$\tfrac{1}{2}\Gamma(\tfrac{1}{2})\sin\,\pi/4 = \int_0^\infty \sin u^2\, du\,. \tag{54}$$

Or

$$\int_0^\infty \sin u^2\, du = \frac{\sqrt{2\pi}}{4}\,, \tag{55}$$

and similarly for the cosine case.

10 The Beta Function – Part 10

The Beta function is defined by

$$B(s_1,s_2) \equiv \int_0^1 u^{s_1-1}(1-u)^{s_2-1}du\,. \tag{56}$$

We begin with the exploratory product of two independent Gamma functions:

$$\Gamma(s_1)\Gamma(s_2) = \int_0^\infty x^{s_1-1}e^{-x}\,dx\int_0^\infty y^{s_2-1}e^{-y}\,dy\,. \tag{57}$$

Next, a change of variables: $x \to u^2$ and $y \to v^2$. Then

$$\begin{aligned}\Gamma(s_1)\Gamma(s_2) &= \int_0^\infty u^{2s_1-2}e^{-u^2}\,2u\,du \int_0^\infty v^{2s_2-2}e^{-v^2}\,2v\,dv \\ &= 4\int_0^\infty u^{2s_1-1}e^{-u^2}\,du \int_0^\infty v^{2s_2-1}e^{-v^2}\,dv \\ &= 4\int_0^\infty \left[\int_0^\infty v^{2s_2-1}e^{-v^2}\,dv\right] u^{2s_1-1}e^{-u^2}\,du \\ &= 4\int_0^\infty \int_0^\infty v^{2s_2-1}e^{-v^2}\,u^{2s_1-1}e^{-u^2}\,dv\,du \\ &= 4\int_0^\infty \int_0^\infty v^{2s_2-1}u^{2s_1-1}e^{-(u^2+v^2)}\,dv\,du \\ &. \end{aligned} \tag{58}$$

Next, one more variable substitution: $v = r\cos\alpha$, $u = r\sin\alpha$, yielding

$$\begin{aligned}\Gamma(s_1)\Gamma(s_2) &= 4\int_0^\infty \int_0^{\pi/2} r^{2s_2-1}(\cos\alpha)^{2s_2-1}r^{2s_1-1}(\sin\alpha)^{2s_1-1}e^{-r^2}\,r\,d\alpha\,dr \\ &= 4\int_0^\infty \int_0^{\pi/2} r^{2s_2-1}r^{2s_1-1}r(\cos\alpha)^{2s_2-1}(\sin\alpha)^{2s_1-1}e^{-r^2}\,d\alpha\,dr \\ &= 4\int_0^\infty \left[\int_0^{\pi/2}(\cos\alpha)^{2s_2-1}(\sin\alpha)^{2s_1-1}d\alpha\right] r^{2s_2-1+2s_1-2}e^{-r^2}\,r\,dr \\ &= \int_0^\infty (r^2)^{s_1+s_2-1}e^{-r^2}2r\,dr \times \int_0^{\pi/2} 2(\cos\alpha)^{2s_2-1}(\sin\alpha)^{2s_1-1}d\alpha\,. \end{aligned} \tag{59}$$

From this we get the simpler form

$$\Gamma(s_1)\Gamma(s_2) = \Gamma(s_1+s_2)\;2\int_0^{\pi/2}(\cos\alpha)^{2s_2-1}(\sin\alpha)^{2s_1-1}d\alpha\,. \tag{60}$$

Hence,

$$\begin{aligned}\frac{\Gamma(s_1)\Gamma(s_2)}{\Gamma(s_1+s_2)} &= 2\int_0^{\pi/2}(\cos\alpha)^{2s_2-1}(\sin\alpha)^{2s_1-1}d\alpha \\ &= \int_0^{\pi/2}(\cos\alpha)^{2s_2-2}(\sin\alpha)^{2s_1-2}(2\sin\alpha\cos\alpha)\,d\alpha\,. \end{aligned} \tag{61}$$

And one last substitution: Let $u = (\sin\alpha)^2$, then

$$B(s_1,s_2) = \frac{\Gamma(s_1)\Gamma(s_2)}{\Gamma(s_1+s_2)} = \int_0^1 u^{s_1-1}(1-u)^{s_2-1}\,du\,. \tag{62}$$

11 The Legendre Duplication Formula – Part 11

This formula looks like the following

$$\Gamma(2s) = \frac{2^{2s-1}}{\sqrt{\pi}}\,\Gamma(s)\Gamma(s+\tfrac{1}{2})\,. \tag{63}$$

We start with Eq. (62) and set $s_1 = s_2 = s$ to get

$$\frac{\Gamma(s)\Gamma(s)}{\Gamma(2s)} = \int_0^1 u^{s-1}(1-u)^{s-1}\,du\,. \tag{64}$$

Not let's make the following variable substitution to the integral on the RHS of this last equation, namely, $u \to \frac{1+x}{2}$, then

$$\begin{aligned}
\text{RHS} &= \int_0^1 \Big(\frac{1+x}{2}\Big)^{s-1}\Big(\frac{1-x}{2}\Big)^{s-1}\frac{dx}{2} \\
&= \frac{1}{2^{2s-1}}\int_{-1}^1 (1-x^2)^{s-1}\,dx \\
&= \frac{1}{2^{2s}}\int_0^1 (1-x^2)^{s-1}\,dx\,.
\end{aligned} \tag{65}$$

But, using the Beta Function, we have that

$$B(\tfrac{1}{2}, s) = 2\int_0^1 (1-x^2)^{s-1}\,dx\,. \tag{66}$$

Hence, from (62)

$$2^{2s-1}\Gamma(s)\Gamma(s) = \Gamma(2s)B(\tfrac{1}{2}, s) = \frac{\Gamma(\frac{1}{2})\Gamma(s)}{\Gamma(\frac{1}{2}+s)}\,. \tag{67}$$

With some manipulation we get

$$2^{2s-1}\Gamma(s)\Gamma(s)\Gamma(\tfrac{1}{2}+s) = \sqrt{\pi}\,\Gamma(2s)\,. \tag{68}$$

And, on solving for $\Gamma(2s)$, we have that

$$\Gamma(2s) = \frac{2^{2s-1}}{\sqrt{\pi}}\,\Gamma(s)\Gamma(\tfrac{1}{2}+s)\,. \tag{69}$$

12 Relation to the Zeta Function – Part 12

Let s be a complex variable. Then the Gamma function of s is given as

$$\Gamma(s) \equiv \int_0^\infty t^{s-1}e^{-t}dt \quad \text{where } \mathrm{Re}(s) > 0\,. \tag{70}$$

What we're about to do is to show a relationship between the Gamma function and the Zeta function. So, let's begin by making a variable substitution in (70), $t \to nu$:

$$\begin{aligned}\Gamma(s) &= \int_0^\infty (nu)^{s-1} e^{-nu} n \, du \\ &= \int_0^\infty n^s u^{s-1} e^{-nu} \, du \,. \end{aligned} \tag{71}$$

Hence,

$$\Gamma(s) \frac{1}{n^s} = \int_0^\infty u^{s-1} e^{-nu} \, du \,. \tag{72}$$

Does the fraction in the LHS look suggestive in the present context? Either way, let's perform a summation on both sides by n, with it going from 1 to infinity:

$$\Gamma(s) \sum_{n=1}^{\infty} \frac{1}{n^s} = \int_0^\infty u^{s-1} \sum_{n=1}^{\infty} e^{-nu} \, du \,. \tag{73}$$

By substituting the zeta function on the left and using the geometric series trick on the right, we get

$$\Gamma(s)\zeta(s) = \int_0^\infty u^{s-1} \left[\frac{1}{1-e^{-u}} - 1 \right] du = \int_0^\infty u^{s-1} \frac{du}{e^u - 1} \,. \tag{74}$$