Word Problems 17: Mixed-Rate Problems #14

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Abstract

In this algebra word problem note, we use the Scheme to solve our fourteenth attempt at what I refer to as a 'mixed-rate problem'.

1 Introduction

Some more tricky problems this time.

2 Word Problem #17.1

We begin with 1 gallon of pure wine in Container 1 and 1 gallon of pure water in Container 2. Now, we take one cup of the wine and add it to the water and let that mixture homogenize. Then we take one cup of that mixture and add it back to the wine. Which container has the highest purity of original contents?[1]

3 Solution 17.1.1: Conceptualizing the Problem

This problem is similar to problem #16.2. First, let's be very clear what we mean by purity of the original content. We mean that we compare the fraction $\frac{\text{wine}}{\text{wine+water}}$ in Container 1 compared to the fraction $\frac{\text{water}}{\text{water+wine}}$ in Container 2.

We have to lead up to the situation in Figure 1 (below). Let c represent the volume of 1 cup in gallons (1/16th gallon). Now, we transfer 1 cup of wine from Container 1 to Container 2, leaving $1 - c$ gallons of wine in Container 1 and $1 + c$ gallons in Container 2. After we leave (or stir) the contents of Container 2 to homogenize, we then take 1 cup of mixture from Container 2 to Container 1, leaving 1 gallon of liquid in Container 1 and 1 gallon of liquid in Container 2.

In Figure 1, we see the values of a and b and we don't know yet what they are, but their fractional ratio a/b will be useful to solve the purity question. The value of k in the figure is equal to b/a.

[1] There are 16 cups to a gallon, though for this problem, I think that the only requirement is that a cup is less than a gallon.

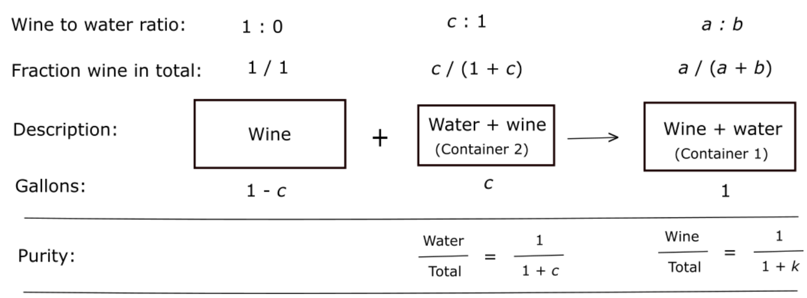

Figure 1. Which is purer: Container 1 (Wine contaminated by water) or Container 2 (Water contaminated by wine)? By the way, the containers are identified by 1 or 2 only in their final states of 'contamination'.

Because the figure has been set up to give us the fractional amounts of wine-to-totals in the containers, we will get the information we need by writing out the conservation equation for wine in this 'before and after' process:

$$(1)(1-c) + \Big(\frac{c}{1+c}\Big)(c) = \Big(\frac{a}{a+b}\Big)(1)\,. \tag{1}$$

4 Solution 17.1.2: Solving the Problem

Now we adopt the usual simplification, $\gamma \equiv a/b$,[2] then (1) becomes

$$(1-c) + \frac{c^2}{1+c} = \frac{1}{1+\gamma^{-1}}\,. \tag{2}$$

But k in the figure is $b/a = \gamma^{-1}$, so

$$(1-c) + \frac{c^2}{1+c} = \frac{1}{1+k}\,. \tag{3}$$

But

$$(1-c) + \frac{c^2}{1+c} = \frac{1-c^2+c^2}{1+c} = \frac{1}{1+c}\,. \tag{4}$$

Plugging this back into (3), we get that $k = c$.

Therefore, Containers 1 and 2 have the same purity, namely

$$\frac{1}{1+c} = \frac{1}{1+1/16} = \frac{16}{17} \tag{5}$$

of their original contents.

[2] I have also used the symbol λ for this fraction, though one isn't better than the other.

5 Word Problem #17.2

Question 174684:[3] 8-year-old Samantha visited Santa at a local department store. He gave her this riddle: “I started working at 15. I spent 1/4 of my working life in a factory. I spent 1/5 of my working life in an office, and I spent 1/3 of my working life as a school caretaker. For the last 13 years of my working life I’ve been Santa Claus. How old am I?” The problem says you need to explain how you got the answer and how you know it is correct.

6 Solution 17.2.1: Conceptualizing the Problem

This problem is easy if we approach it with the two questions: Are there any totals? Are there any parts? In fact, there are both.

$$\begin{aligned}(\text{Santa's total life time}) = {} & (\text{Santa's time prior to working}) \\ & + \sum (\text{Santa's time working at various jobs})\,. \end{aligned} \tag{6}$$

How do I know that this equation is correct so far? I know because the RHS is the sum of a collection of collectively exclusive, mutually exclusive time intervals, whatever their individual natures are.

Let’s let A stand for ‘Santa’s total life time’, meaning, his current age. Let W stand for Santa’s working life $= A -$ time Santa didn’t work $= A - 15$. The rest of the parts of (6) are given to us as fractional amounts of W or as a specific number:

$$W = A - 15\,, \tag{7a}$$

$$A = 15 + \frac{1}{4}W + \frac{1}{5}W + \frac{1}{3}W + 13\,. \tag{7b}$$

which has solution $A = 75$ years old.

7 Word Problem #17.3

Two small pitchers and one large pitcher can hold 8 cups of water. One large pitcher minus one small pitcher constitutes 2 cups of water. How many cups of water can each pitcher hold?[4]

Comment: This problem is similar to the ‘heat-loss problem through glass and plaster’ (#16.1), and also to the ‘shoes and boots price per pair in the footware problem’ (#10.3).

[3]Found at https://www.algebra.com/algebra.
[4]Found at http://regentsprep.org/Regents/math/ALGEBRA/AE3/PracWo

8 Solution 17.3.1: Conceptualizing the Problem

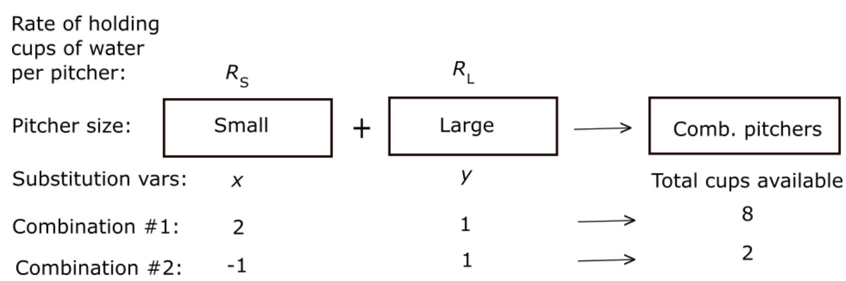

Figure 2. Standard setup for calculation: $R_S x + R_L y =$ Total cups.

We can easily glean from Figure 2 the information to write the coupled equations

$$\begin{aligned} (2)R_S + (1)R_L &= 8\,, \\ (-1)R_S + (1)R_L &= 2\,, \end{aligned} \tag{8}$$

which as solution $R_S = 2$ cups per small pitcher, and $R_L = 4$ cups per large pitcher.

9 Word Problem #17.4

Hard algebra word problems?[5] Hockey teams' receive 2 points when they win and 1 point when they tie. One season, a team won a championship with 56 points. They won 10 more games than they tied. How many wins and how many ties did the team have?

10 Solution 17.4.1: Conceptualizing the Problem

This is almost a 'standard' mixed-rate problem. In a standard mixed-rate problem, all necessary information is presented either as invariants in a before-and-after process or as relations between totals and their parts. In a nonstandard mixed-rate problem, some necessary information is given in some other form. I refer to that other form as *constitutive*, meaning that it constitutes an essential part of the system that is to be solved, but is given in nonstandard form. However, I don't want to give the impression that constitutive information is inferior. The only eason I justify referring to a class of mixed-rate problems as 'standard' is only due to the prevelence of the form I find them in.

Let W be the number of wins and T be the number of ties.

[5] Found at https://answers.yahoo.com/question/index?qid=20080309104615AAMKboY.

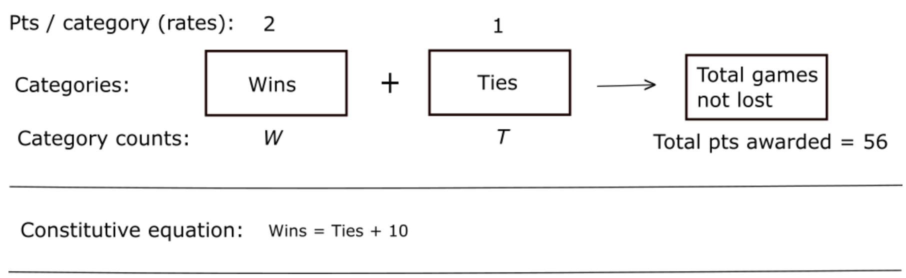

Figure 3. Standard and nonstandard information given to us.

The standard information given to us is in the form of a total equal to the sum of its parts:

$$(2)W + (1)T = 56\,. \tag{9a}$$

We were also given the constitutive relation

$$W = T + 10\,. \tag{9b}$$

These equations have the solution $T = 12$ and $W = 22$.

Before I leave this issue of constitutive information, I want to underscore the 'just so happens' nature of the information we were given: *They won 10 more games than they tied*. The problem could have been rewritten to claim that *It just so happens that they won 10 more games than they tied.* This information could have also been given as *It just so happens that* (their wins)^2 − (their ties)^2 = 340, which gives the right answers, so long as we ignore the extraneous roots. We could also claim that *It just so happens that the average of their wins and their ties is 17.*

11 Conclusion

Problem #17.1 above is one of the most difficult problems I've solved so far in this series of notes on word problems. My copy of my first attempt at it had a poorly constructed figure and three pages of confusing summations. What's worse, for all that apparent effort, I didn't find the solution. The reason I was able to solve it this time was because I focussed on totals and their parts, and process invariants, and making a very good figure.

It's fashionable these days to skip the figure and replace it with a tablular version of its information. I'm sorry, but I cannot get the Big Picture from a table, no matter how well crafted it is. I need the visual aid of the figure to show me how this problem is similar to others I have already done, and to make use of their flexibility in presenting data that comes in many forms.