Word Problems 25

P. Reany

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Abstract

In this algebra word problem note, we use the Scheme to solve a variety of problem that may or may not include 'mixed-rate' types.

1 Introduction

We venture to all kinds of words problem for my foreseeable future, beginning with the so-called 'age problem' type.

2 Word Problem #25.1

[1] Cary is 9 years older than Dan. In 7 years, the sum of their ages will equal 93. Find both of their ages now.

3 Solution 25.1.1: Conceptualizing the Problem

Age problems tend to follow a definite pattern: Give a relation between (among) two or more things or people **now** and then one or more relations how they relate in the past or future or both. Let C stand for Cary's age now and D for Dan's age now. In the following figure, I show how I like to deal with this sort of problem.

Now	Seven years from now
$C = D + 9$	$(C + 7) + (D + 7) = 93$

Figure 1. The equations are labeled with their 'time stamp' on them.

The solutions to these equations are $C = 44$ and $D = 35$.

[1] Found at https://www.algebra.com/algebra/homework/word/age/Solving-Age-Problems.lesson.

4 Word Problem #25.2

Starting with x amount of pure liquid A in a beaker, we will n times repeat the following process: Draw out of the beaker y amount of the liquid (by volume) and add y amount of a different liquid Z to the beaker and let the contents come to a homogeneous mixture before repeating the cycle. Show that, at the nth cycle, the amount of the original liquid A is given by the formula

$$Q_n = x(1 - y/x)^n \,. \tag{1}$$

5 Solution 25.2.1: Conceptualizing the Problem

I'll use a standard induction proof for this formula, but before I get to that, I want to first motivate the formula.

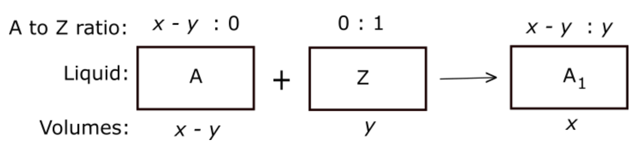

Figure 2. We'll consider the amount of liquid A in the beaker after the first cycle. The beaker state on the left shows the volume after the y quantity has been removed, but on the right, after the y amount of liquid Z has been added.

The amount of liquid A in mixture A_1 is given by

$$Q_1 = \frac{x - y}{x}(x) = x - y = x(1 - y/x)\,. \tag{2}$$

Let's do one more cycle. We now pour off y volume of liquid in the beaker.

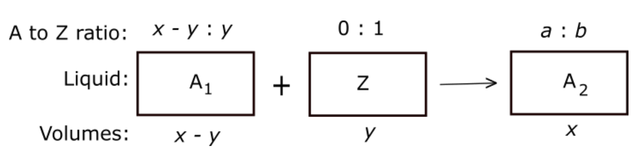

Figure 3. We're now on the second cycle.

The quantity of A in mixture A_2 is given by

$$Q_2 = \frac{a}{a + b}\,(x)\,. \tag{3}$$

To get $a/(a + b)$ we can balance this cycle for liquid A:

$$Q_2 = \frac{x - y}{x}(x - y) = \frac{a}{a + b}\,(x)\,. \tag{4}$$

That was easy. We can ignore a and b. Also, we just need to conform Q_2 to the pattern we're looking to prove.

$$Q_2 = \frac{x-y}{x}(x-y) = x(1-y/x)^2\,. \tag{5}$$

Now begins the induction proof of the formula (1). Clearly, the formula holds for $n = 0$, the base step. Next, we take as the inductive hypothesis, the assertion that

$$Q_k = x(1-y/x)^k\,, \tag{6}$$

and show that it holds for step $k+1$. That is, the last equation is still true if we replace k by $k+1$.

Now, at the end of the kth cycle, we have x amount of liquid, divided between Q_k amount of the original liquid A (by the inductive hypothesis) and the rest is liquid Z, of amount $x - Q_k$. This is presumed to be a homogeneous mixture, and then we draw off y amount from the beaker, and we're ready for the next step: the cycle that takes us from k to $k+1$:

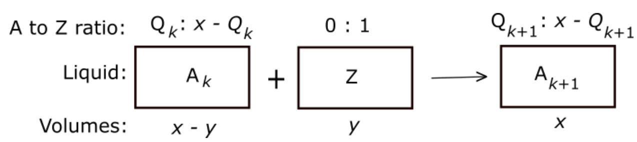

Figure 4. We're now on the $(k+1)$st cycle.

To recap: the end of the kth cycle left us with x amount of liquid in the beaker with Q_k amount of liquid A in liquid A_k. The starting point of the beaker in Figure 4 is right after we removed y amount of liquid A_k, leaving $x - y$ amount of liquid in the beaker.

So, balancing on liquid A across the cycle, we get

$$\frac{Q_k}{x}(x-y) + 0 = \frac{Q_{k+1}}{x}x\,. \tag{7a}$$

Simplifying, we get

$$Q_{k+1} = Q_k(1-y/x)\,. \tag{7b}$$

Using the inductive hypothesis (6) to substitute in for Q_k, we get

$$Q_{k+1} = x(1-y/x)^k \cdot (x-y) = x(1-y/x)^{k+1}\,. \tag{8}$$

And that's what we had to show.

6 Word Problem #25.3

[2] The ratio by volume of soap to alcohol to water in a certain solution is 2 : 50 : 100. After the solution is altered so that the ratio of soap to alcohol is doubled,

[2]Found at https://gmatclub.com/forum/the-ratio-by-volume-of-soap-to-alcohol-to-water-in-a-68933.html.

and the ratio of soap to water is halved, there is 100cc of alcohol. How many cc's of water does this new solution contain?

7 Solution 25.3.1: Conceptualizing the Problem

What does it mean to double or half a ratio? To double the ratio $a : b$ is to yield $2a : b$. To half the ratio $d : e$ is to get $d : 2e$. So, we deal with these changes in steps.

Step 1) Soap to alcohol is doubled means we get the ratios $4 : 50 : 100$.

Step 2) As we left things in the last step, we also doubled ratio of soap to water. To half the ratio of soap to water means $2 : 50 : 100 \rightarrow 2 : 50 : 200$. To combine these two effects, we can either half the fifty in $2 : 50 : 200$ to get $2 : 25 : 200$ or quadruple the 100 in $4 : 50 : 100$ to get $4 : 50 : 400$. Either way, both alterations are in effect.

The question of how much water is obtained is given by the proportion

$$\frac{400}{50} = \frac{x}{100}\,, \tag{9}$$

with answer $x = 800$cc.

8 Word Problem #25.4

[3] At a certain company, 40% of the women employees and 50% of the men employees are 50 or older. If that amounts to 42% of all the company's employees are 50 or older, what percentage of the company's employees are men?

9 Solution 25.4.1: Conceptualizing the Problem

My experience with percentage problems has led me to start with the precise definition of the percentage I'm tasked with finding and then, if useful, search for totals and parts. But often the information needed is provided in constitutive relations, instead.

Let's begin with the initial percentages. Let M be the number of men in the company. Let W be the number of women in the company.

$$.40W = W_{\geq 50} \quad \text{and} \quad .50M = M_{\geq 50}\,. \tag{10}$$

The percentage we're looking for is given by

$$P = \frac{M}{M+W} \times 100\%\,. \tag{11}$$

Let's begin with a figure.

[3]Found at https://gmatclub.com/forum/at-a-certain-company-40-of-the-women-are-over-50-years -old-and-50-o-205994.html.

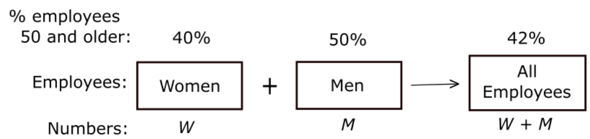

Figure 5. I was raised on the commercial: Things go better with Coke (the soft drink), but as a algebraist, I believe that things go better with diagrams.

Now, balancing on the number of employees 50 and older on both sides, we get:

$$.40W + .50M = .42(W + M)\,. \tag{12}$$

My favorite way to proceed in this sitution is to divide through by M and let $\lambda = W/M$. Then (11) and (12) become

$$P = \frac{1}{1+\lambda} \times 100\%\,, \tag{13a}$$

and

$$.40\lambda + .50 = .42(\lambda + 1)\,. \tag{13b}$$

Solving this last equation for λ, we get $\lambda = 4$. Substituting this into (13a), we get

$$P = \frac{1}{1+4} \times 100\%\,. \tag{14}$$

From this we get that 20% of the company's employees are men.

10 Conclusion

Well, that's *another* fine mess we've gotten ourselves out of by the skin of our teeth.