Word Problem 2: Coins in a jar

P. Reany

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Abstract

In this algebra word problem, we use the Scheme to solve the 'coins in a jar' word problem, and introduce a couple heuristics we didn't introduce the last time. We also go over conversion factors and units.

1 Introduction

We will encounter a couple more heuristics this time that were not mentioned in the last paper. Ouch! Not more rules? Well I, for one, am glad to have them. Our first one is the following:

Zeroth Rule of Problem Solving

Make any assumption necessary in order to solve the problem in a reasonable amount of time with a reasonable amount of effort.

Back when I was taking a class in programming, the instructor advised us to always make our assumptions explicit, not only to prevent problems, but also to find them if they occur unexpectedly. If you find yourself frustrated while troubleshooting your faulty solution to a problem, you may be stymied because the source of the trouble may lie in one of those assumptions you didn't bother make explicit in the formal presentation of the solution.

The top-down approach

Our next heuristic is also inspired by my formal education in computer programming: The top-down approach. We think of solving a word problem as first translating the word problem from English into algebraic form and then solving the system of equations (or inequalities) thus derived. The *art* of solving word problems is not in the algebra per se, because that is just a matter of learning techniques that anyone can memorize and master with some practice. The art comes from converting the English words into the algebra. This is the technique I like to use: Find the totals and parts and/or the invariants, etc, and then employ them in English sentences that are complete but don't immediatley appear as 'algebraic'. Then refine the sentences step-by-step until the end result is an algebraic equation or inequality. Then solve the system.

The tendency of the novice problem solver is to make big steps, mangling together many unclear concepts into one broken result. Use simple refinements each step as you go. To quote from the movie *Contact*, "Small moves."

2 Word Problem

A jar containing nickels and dimes has \$1.05 worth of coins in it. If the jar contains exactly 16 coins, how many are nickels and how many are dimes?

Now, as it strands, this problem is ambiguous. What we need to know is if there are *only* nickels and dimes in the jar. By Zeroth Rule (above) then it is reasonable to assume that there are only nickels and dimes in the jar.

3 Solution Part 1: Conceptualizing the problem

As we claimed (without proof) last time, most problems will contain at least one total, so, generally speaking, unless there is an obvious reason *not* to start our solution by looking for a total, let's do so.[1]

There are two obvious totals in the given information. The first is the total money in coins, being \$1.05. The second is the total number of coins in the jar. Now, this is where the assumption that there are only nickels and dimes in the jar comes in handy. You see, our procedure is to find a total, discover all of its parts, add those parts together, and set that sum equal to the total. Since we assume that the 'coin parts' exist only as nickels and dimes, we begin our formulation of this equation in the simple, easy-to-understand form of

$$(\text{dollar value in nickels}) + (\text{dollar value in dimes}) = \$1.05\,. \tag{1}$$

As part of our conceptualization of this process of sorting the coins by type, we we can abstract this invariant process in the figure below.

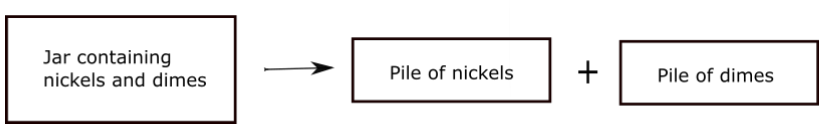

Figure 1. This graphic represents our imagined sorting of all the coins into a pile of nickels and a pile of dimes, leaving invariant the number of each.

[1] When searching for 'parts', we need to find enough of them to add up to the total we seek. However, we need to be sure that the parts are mutually exclusive (they don't intersect) so that we don't exceed the total. This is what is meant by the expression 'mutually exclusive and collectively exhaustive'.

Now, before we make our first step-wise refinement of Eq. (1), let's ask a more general question in preparation. What does it mean to calculate the dollar value of a pile of coins of a single type? It means to count the number of coins and then multiply this number by the dollar value of each coin:

$$(\text{value of a single coin})(\#\text{coins}) = \text{value of pile of coins}\,, \tag{2}$$

where we have placed the conversion factor to the left of the number of coins, which is customary but not necessary.[2]

Time out, please!
A *conversion factor* is a rate of change; specifically, the rate of change of things in one unit (in the numerator) into things into some other unit (in the denominator), and vice versa. Perhaps you think that it's more proper to restrict our notion of a conversion factor to converting between things of 'like nature', such as in the case of converting between inches to feet or yards to meters — all dealing with lengths or distances in this particular example. But this is an unhelpful and unnecessary restriction. What we really want is to form a conceptual basis for solving algebra problems in which the leastest number of primitive notions conceivable can cover the most number of particular cases.

Let's consider two other examples. First, velocity. Velocity is indeed a *rate of change*, but it is also a conversion factor, changing timelike things into distancelike (or spacelike) things.

Second, one of my favorite examples of both 'totals being the sum of their parts' and the use of conversion factors rolled into the familiar example of what we owe on our grocery purchases: the total cost of groceries. To simplify matters, we'll assume we're buying at the grocery store two types of untaxed groceries and paying with cash. Suppose we are buying four of one kind of apple at \$0.50/apple and three cans of peas at \$1.14/can. Again, we begin with a 'total' equation:

$$\text{total grocery bill in }\$ = (\$\text{ cost of apples}) + (\$\text{ cost of cans of peas})\,. \tag{3a}$$

In the past note I claimed that the algebra problem solver will have to know how to deal with units, and here is a good place for us to show this explicitly. A stepwise refinement of this last equation, gives

$$\begin{aligned}\text{total }\$\text{ grocery bill} &= \frac{\$0.50}{\text{apple}}(4\text{ apples}) + \frac{\$1.14}{\text{can of peas}}(3\text{ cans})\\ &= \$2.00 + \$3.42\,,\end{aligned} \tag{3b}$$

where each of these last two terms is called a *subtotal*. And the total cost of our groceries is \$5.42.

At the conceptual level, what is going on here? The conversion factors are telling us how much (many) goods we can take from the store converted into how much cash we must leave at the register.

[2]In fact, in stoichiometry (chemistry) the practice is to successively pile on conversion factors on the right.

Returning to our coin problem, the value of a single unspecified type coin is $\dfrac{\$\text{X.YZ}}{1\text{ coin}} = \dfrac{\$\text{X.YZ}}{\text{coin}}$, dropping the superfluous 1 in the denominator. Now, just to be a bit exotic, let's say the coin in question is a \$20 gold piece, and we have twenty of them. Then Eq. (2) becomes

$$\left(\frac{\$20.00}{\text{coin}}\right)(20\,\text{coins}) = \$400.00\,. \tag{4a}$$

In the language of 'units' in algebra, we say that in the above equation the coin unit has 'cancelled out'. We could have made this more explicit by writing

$$\left(\frac{\$20.00}{\cancel{\text{coin}}}\right)(20\,\cancel{\text{coins}}) = \$400.00\,. \tag{4b}$$

Time in. (Thanks for your patience!)

Our first **step-wise refinement** on Eq. (1) yields

$$(\$\text{ value of a nickel})(\#\text{nickels}) + (\$\text{ value of a dime})(\#\text{dimes}) = \$1.05\,. \tag{5}$$

We still have not yet introduced any variables in this algebra problem. We could have at the start, and it wouldn't have hurt, but it wouldn't have helped much either. Let's introduce them now, setting $D = \#\text{dimes}$ and $N = \#\text{nickels}$. Then, for our next **step-wise refinement** we get

$$\left(\frac{\$0.05}{\text{nickel}}\right)(N\text{ nickels}) + \left(\frac{\$0.10}{\text{dime}}\right)(D\text{ dimes}) = \$1.05\,. \tag{6}$$

Thus, we have two unknowns but only one equation. So, we need one more equation in N and D to be able to solve for the two unknowns. Now, what I am about to write may seem terribly pedantic, but if we were writing these equations in a computer language with strong typing requirements, we would have to pay very close attention to the units of our subtotals. We actually did that properly when we considered the units in the subtotals of the 'value' equations above. But now it's time to write down the total coins equation:

$$(\#\text{ nickel coins}) + (\#\text{ dime coins}) = \text{total coins} = 16\text{ coins}\,. \tag{7}$$

In other words, coins + coins = coins.

4 Solution Part 2: Solving the system

Fortunately, I don't intend to be this pedantic about units in future word problems, but I did want to be very clear about the meaning of 'adding subtotals to get a total' once. Now, I'll strip the equations of *all* units and write

$$1.05 = 0.05N + 0.10D\,, \tag{8a}$$
$$16 = N + D\,. \tag{8b}$$

This system has the unique solution $N = 11$ and $D = 5$. Thus, there are 11 nickels and 5 dimes in the jar.

Perhaps you're wondering why I didn't use x and y instead of N and D. Mathematically speaking, the choice of variable identifiers is arbitrary, so long as they can be distinguished. A more serious issue to deal with is the rcommendation of some authors to choose the two variables as x and $16 - x$, what I refer to as 'accelerated substitution'. Although I can see the advantage of this for some people in some problems, I find it a bad habit, especially for the novice. I'm trying to teach a technique for solving algebra word problems that won't break down when the number of variables increases beyond two, and we'll be getting to those problems soon enough.

5 Conclusion

I've gone to great detail to explain why I made the steps and choice of notations that I did in the solving of the coin problem, but I won't be doing similarly much in the future. As for the weakness of using accelerated substitution in algebra word problem as the number of unknowns goes beyond two, the reader will see for him or herself soon enough.