Word Problems 33: Kinematic, Ratio, and Mixed-Rate Problems

P. Reany

November 16, 2024

Abstract

In this algebra word-problem note, we use the Scheme to solve the interesting problem that involves distance, velocities, and times — and arbitrary units of distance. We also have to deal with some 'specialty' problems, unlike those we have dealt with before.

1 Word Problem #33.1

A thief is 20 steps ahead of a cop, and takes 6 steps while the cop takes 5, but 5 of the cop's steps equal 8 of the thief's; how far will the thief run before he is caught?
[This is Question #204111 from **www.algebra.com**.]

2 33.1.1: Conceptualizing the Problem

To begin with, it's clear that the pertinent variables of this problem are distance, velocity, and time. Written in terms of distance, we usually write:

$$D = v \times t\,. \tag{1}$$

Let's visualize this through use of a distance diagram.

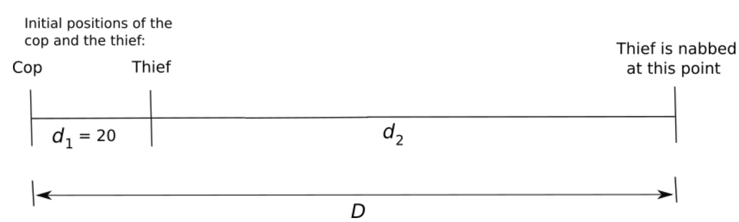

Figure 1. We conceive of the distance the cop traveled as the sum of the distance he had to reach the thief's starting point, which is d_1, and then

the additional distance that the thief ran, which is d_2.

We can begin with an equation of the form: the total is equal to the sum of its parts. From Fig. 1, that gives us:

$$D = D_c = d_1 + d_2 = 20 + d_2\,, \tag{2}$$

where D_c is the distance the cop must run.

At this point we have to make some simplifying assumptions. First, that the stride length for both cop and thief are the same all the time. Second, we aren't given the stride length for either the cop or the thief, so we need to invent a convenient unit of length. For this, I chose the running stride-step length of the thief to be called one *leap.* I'll also assume that their velocities are constant throughout.

3 33.1.2: Solving the Problem

So, what are we to make about the claim that the thief is twenty steps ahead of the cop at the start? Are these steps in thief units or cop units? We have to decide. I choose to regard them a thief step units.

Next, we ask ourselves how we are to relate the variables of the cop to those of the thief. We discover this by the process of elimination. First, do they run the same distance? No. Do they run at the same velocities? No. Do they run for the same amount of time? Yes. Then that's where we begin. Let T_c be the time the cop ran, and let T_t be the time the thief ran. So, we begin our equations with this:[1]

$$T_c = T_t\,. \tag{3}$$

Solving for t in (1) and substituting into (3), we get

$$\frac{D_c}{v_c} = \frac{D_t}{v_t}\,. \tag{4}$$

This then becomes

$$\frac{20 + d_2}{v_c} = \frac{d_2}{v_t}\,. \tag{5}$$

But this can be rewritten as

$$20 + d_2 = \frac{v_c}{v_t}\,d_2\,. \tag{6}$$

Next, we need to reason out the ratio v_c/v_t. Common sense tells us that this ratio had better be greater than unity. So, let Δt be the time it takes for the thief to run his 6 leaps, and that is the same time it takes the cop to run his 5

[1]I'm being somewhat prejudiced in my approach to choosing which variable to begin with. The truth is that the distances of both runners are related, as well as their velocities are related to each other. I'm starting with the time variable only because they are equal, but one could start with any of the variables.

strides. (We don't actually care what the numerical value of Δt is.) Therefore, with 1 cop step equal to 8/5 leap,

$$\frac{v_c}{v_t} = \frac{(5 \text{ cop steps})/\Delta t}{(6 \text{ leaps})/\Delta t} = \frac{5 \text{ cop steps}}{6 \text{ leaps}} = \frac{5\left(\frac{8}{5} \text{ leap}\right)}{6 \text{ leaps}} = \frac{4}{3}\,. \tag{7}$$

On substituting this result into (6) and solving for d_2, we get

$$d_2 = \frac{20}{4/3 - 1} = \frac{20}{1/3} = 60 \text{ [leaps]} = 60 \text{ [thief steps]}\,. \tag{8}$$

4 Word Problem #33.2

A modernistic painting consists of triangles, rectangles, and pentagons, all drawn so as to not overlap and to not share sides. Within each rectangle are drawn two red roses, and each pentagon contains 5 carnations. How many triangles, rectangles, and pentagons are there in the painting if the painting contains 40 geometrical figures, 153 sides of the geometrical figures, and 72 flowers?
[I don't recall where I got this problem from, but it's labeled #51, page 193.]

5 33.2.1: Conceptualizing the Problem

To begin with, all we need to do is to place all the relevant information in one of our standard, low-cost diagrams, such as below.

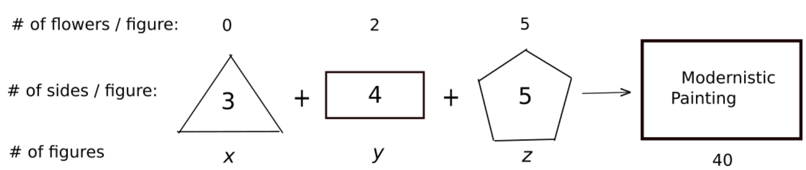

Figure 2. It's all in setting-up the data efficiently, visually. Slow and steady wins the race. Every total is the sum of all its parts!

6 33.2.2: Solving the Problem

From the total number of figures, we get the equation:

$$x + y + z = 40\,. \tag{9}$$

From the total number of sides, we get the equation:

$$3x + 4y + 5z = 153\,. \tag{10}$$

From the total number of flowers, we get the equation:

$$0 + 2y + 5z = 72\,. \tag{11}$$

Solving for these unknowns, we have that

$$x = 13\,, \quad y = 21\,, \quad z = 6\,. \tag{12}$$

7 Word Problem #33.3

Question 87072 It takes Mars 1.88 years to orbit the sun, and Mercury, 0.24 years. If they are currently aligned with the sun, what is the shortest (positive) time from now until they are similarly aligned.

8 33.3.1: Solving the Problem

If it isn't obvious to the reader that we should first remove the decimal points, it should be soon. To that end, we let

$$1.88 \text{ years} \longrightarrow 188 \text{ centi-years}\,, \tag{13}$$
$$0.24 \text{ years} \longrightarrow 24 \text{ centi-years}\,. \tag{14}$$

So, we are looking for the shortest positive time Δt such that there exists an n and an m, such that

$$\Delta t = n \times 188 = m \times 24\,. \tag{15}$$

Then, of course,

$$\Delta t = \mathrm{LCM}(188, 24) = 1128\,, \tag{16}$$

where our answer is in centi-years. We then divide this by 100 to return to the unit of years, which gives us 11.28 years.

9 Word Problem #33.4

Question 123158 A forrester has a mixture of gasoline and oil in ratio 32 : 1 (gas : oil). He wants to produce two gallons of mixture in ratio 40 : 1. How much of the original mixture and pure gasoline needs to be added together to get this desired mixture?

10 33.4.1: Conceptualizing the Problem

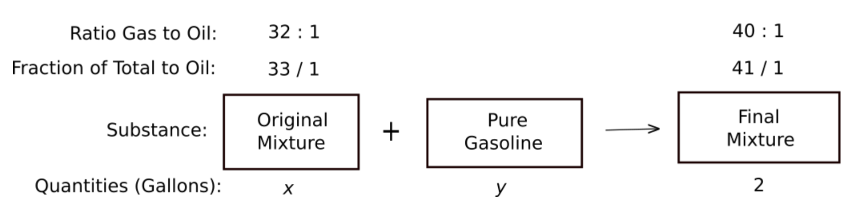

Figure 3. Half of being successful is just to be organized. All we need to do now is to balance on the quantities.

11 33.4.2: Solving the Problem

Balancing on the oil, we have:

$$\frac{1}{33}x + 0 \cdot y = \frac{1}{41}2\,, \tag{17}$$

which gives us $x = 1.6$. The $y = 2 - 1.6 = 0.4$ in gallons.

12 Word Problem #33.5

Question 482126 Machine A can sort products twice as fast as Machine C, and Machine B can sort products at a rate of 300 products fewer per minute than Machine C. When the three machines work together, they have an average sorting rate of 2150 products per minute. How many product sorts (ps) per minute does Machine C make?

13 33.5.1: Conceptualizing the Problem

Clearly, this is a mixed-rate problem: 3 machines are working together to complete one job. Every total is the sum of its parts, thus,

$$1 \text{ job} = \big(\text{PJDB A}\big) + \big(\text{PJDB B}\big) + \big(\text{PJDB C}\big)\,, \tag{18}$$

where 'PJDB' means 'part of the job being done by'. Now, the amount of the job which is done by a particular generic machine Y is the product of machine's rate R_Y at which it performs the job, times T, the time the machine is at work. And, although our three machines each work at their own rates, they each work over the same period of time T. Thus, (18) becomes the generic relation:

$$1 \text{ job} = \big(R_{\text{A}} + R_{\text{B}} + R_{\text{C}}\big)T\,, \tag{19}$$

after we factored out the common time T, and where the units of the R's is 'job per unit time'. To proceed, we need to replace these units by ps per unit time, and to replace the '1 job' on the left by 'Z ps', giving us[2]

$$Z\ [\,\text{ps}\,] = \big(R_{\text{A}} + R_{\text{B}} + R_{\text{C}}\big)T\,. \tag{20}$$

14 33.5.2: Solving the Problem

If we now divide (20) through by T, we get the rate at which the 3 machines, working together, make product sorts, and Z/T becomes the number of product sorts per minute. Therefore,

$$2150 \text{ ps/min} = R_{\text{A}} + R_{\text{B}} + R_{\text{C}}\,. \tag{21}$$

[2]Actually, we don't care what either Z or T is by themselves, but we do care what Z/T is because that is the given value 2150 ps/min.

But, $R_{\mathrm{A}} = 2R_{\mathrm{C}}$ and $R_{\mathrm{B}} = R_{\mathrm{C}} - 300$ ps/min. Hence,

$$2150 \text{ ps/min} = 2R_{\mathrm{C}} + (R_{\mathrm{C}} - 300 \text{ ps/min}) + R_{\mathrm{C}} \,. \tag{22}$$

On solving this for R_{C}, we get

$$R_{\mathrm{C}} = 612.5 \text{ ps/min} \,, \tag{23}$$

which is the rate at which Machine C makes product sorts.

15 Word Problem #33.6

Question: One liter of salt water is 0.5% salt. How much water must be evaporated from this solution so that the result is 2% salt solution?

16 33.6.1: Conceptualizing the Problem

When we talk about a percentage of a substance, we mean to adopt the same units as the thing we are taking the percentage of. In problems such as we have encountered in this series, it is either percentages by volume or by weight (mass). But how are we to know which to choose in a given problem, if it's not specified? When we mix together two or more different volumes, it's generally safe to assume that the volumes are conserved.

If I mix cream with my coffee, I can safely assume that the volume of my coffee afterwards is the sum of the volumes of the initial coffee plus the volume of the cream. But what about adding a teaspoon of sugar to my coffee? First, how do I accurately measure the 'volume' of the sugar for scientific, not culinary, purposes? I don't know. But one thing is sure, the masses of quantities involved will always be conserved.

Therefore, I will interpret the '0.5% salt' to be a percentage by weight (mass). In this particular problem, I will assume that the quantities of salt dissolved in the water will not appreciably affect the volume of the fluids. Therefore, the water volume is conserved in this before-and-after process.

17 33.6.2: Solving the Problem

In Fig. 4, we see that the percentages have been interpreted as by weight (mass), not by volume.

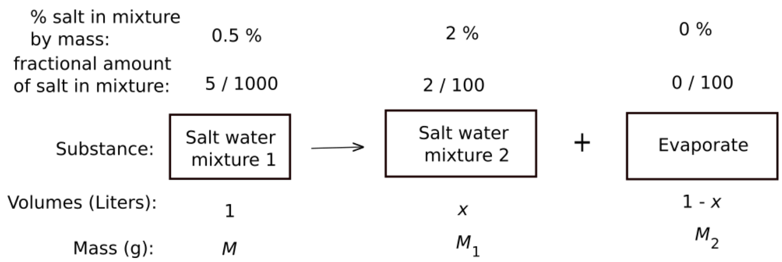

Figure 4. The percentages have been interpreted as by weight (mass), not by volume. The conservation of volume has already been accounted for.

By balancing on the salt throughout this process, we get

$$\frac{5}{1000}M = \frac{2}{100}M_1 + 0\,. \tag{24}$$

The mass of the original mixture is equal to the mass of 1 liter of water (1000 g) plus the mass of the salt in the mixture:

$$\text{mass of salt} = \frac{5}{1000}M = \frac{5}{1000}1000\text{g} = 5\,\text{g}\,. \tag{25}$$

Therefore,

$$M = 1005\,\text{g}\,. \tag{26}$$

From (24) and (26), we have that

$$M_1 = .25M = .25 \times 1005\text{g} = 251.25\text{g}\,. \tag{27}$$

So,

$$M_2 = M - M_1 \approx 754\,\text{g}\,. \tag{28}$$

If we convert this to liters, we find that 0.754 liters of water must be evaporated.

18 Word Problem #33.7

Question (source unknown): A 10 liter mixture of cranberry juice contains juice and water in ratio 3 : 2. Half of the mixture is removed and then 5 liters of pure cranberry juice is added in its place. If this same procedure is repeated once more, what will be the ratio of juice to water in the final mixture?

a) 5 : 3

b) 6 : 4

c) 8 : 2

d) 17 : 3

e) 9 : 1

19 33.7.1: Conceptualizing the Problem

We'll work this out in two steps. First, we don't care what happens to the 5 liters of mixture that is removed. All we care about it that the ratio of juice to water in the remaining mixture is the same before and after the removal.

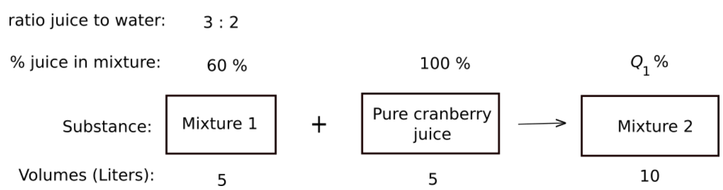

Figure 5. The fractional amount of juice to liquid in Mixture 1 is 3/5, which is 60%.

We can think of Mixture 1 as being built up in units of 5 parts at a time. Three of those parts is juice and two of those parts is water. Therefore, the fractional amount of juice in the mixture is 3/5, which is 60%.

By the way, we are conceptualizing all ratios and percentages in this problem as by volume.

20 33.7.2: Solving the Problem

Our next step is to balance on juice (using Fig. 5):

$$(.6)(5) + (1.00)(5) = \frac{Q_1}{100}(10)\,, \tag{29}$$

which gives us Q_1 as 80, or 80%.

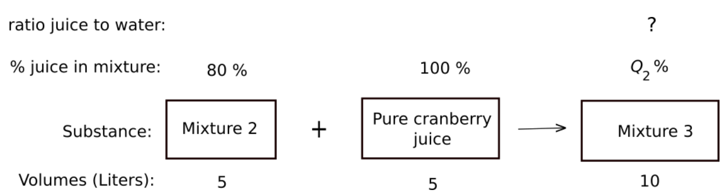

Figure 6. From our previous calculation, the percentage of juice to water in Mixture 2 is 80%. What we're looking for is the ratio of juice to water in Mixture 3.

Again, we balance on juice (using Fig. 6):

$$(.8)(5) + (1.00)(5) = \frac{Q_2}{100}(10)\,, \tag{30}$$

which gives us Q_2 as 90, or 90%.

Now, to find the final ratio of juice to water in Mixture 3, we need to reverse the process and go from percentage to fractional amount to the final ratio. So, 90% is a fractional amount of 9/10, which means that we have 9 parts juice to 10 parts juice + water. Then, the part that is water (out of the ten parts) is one part. Therefore, the final ratio of juice to water is 9 : 1. Thus, we choose answer e) above.

21 Word Problem #33.8

Problem (from https://gmatclub.com/): Every student at Darcy School is a member of one or more school clubs, which we'll refer to as Clubs A,B,C. The number of students in exactly two clubs to those in just one club is 4 : 3. The ratio of the number of students in exactly two clubs to the number of students in two or three clubs is 5 : 7. Which of the following choices could be the number of students in Darcy School?

a) 63

b) 69

c) 74

d) 82

e) 86

22 33.8.1: Conceptualizing the Problem

The most important step is conceptual: We need to realize that, from the given information, we know that the set of Darcy students is partitioned by club membership. So, let x be the number of students in just one club; let y be the number of students in exactly two clubs; and let z be the number of students in exactly three clubs.

Club no. of memberships	1 Club	2 Clubs	3 Clubs
Variable Names	x	y	z

Figure 7. Every student is in one and only one of these divisions.

23 33.8.2: Solving the Problem

According to the given information and from Fig. 7, it's clear that the total number of students, T, at Darcy is given by

$$T = x + y + z\,. \tag{31}$$

But before we can use this equation, we must first use the given information in the form of ratios.

So, we receive the given information that the number of students in exactly two clubs to those in just one club is $4:3$ as

$$y : x :: 4 : 3\,, \tag{32}$$

to use 'modern' notation. But we can make this more 'algebraic' by putting it in the form

$$\frac{y}{x} = \frac{4}{3}\,, \tag{33}$$

from which we get that

$$y = \frac{4}{3}x\,. \tag{34}$$

Lastly, we're told that the ratio of the number of students in exactly two clubs (y) to the number of students in two or three clubs ($y + z$) is $5 : 7$. Hence,

$$y : (y + z) :: 5 : 7\,, \tag{35}$$

to use 'modern' notation. Again, we can make this more 'algebraic' by putting it in the form

$$\frac{y}{y+z} = \frac{5}{7}\,, \tag{36}$$

from which we get that

$$y = \frac{5}{7}(y + z)\,. \tag{37}$$

Solving this for z, we have that

$$z = \frac{2}{5}y = \frac{2}{5} \times \frac{4}{3}x = \frac{8}{15}x\,. \tag{38}$$

On substituting the results of (34) and (38) into (31), we get

$$T = x + \frac{4}{3}x + \frac{8}{15}x = \frac{43}{15}x\,. \tag{39}$$

Therefore, whatever the value of T is, it must be divisible by the prime number 43. Of the possible answers given, only 86 fits that constraint, so the answer is e).

Second solution:

In general, if three positive numbers, say a, b, and c, sum to a total T, then, given $a : b$ and $b : c$, we can find the proper ratio $a : c$. In our problem, we know that

$$\begin{aligned} x : y &: z \\ 3 : 4 &: \\ : 5 &: 2\,. \end{aligned}$$

To match these, we need the LCM(4,5), which is 20, we gives us

$$\begin{aligned} x : y &: z \\ 15 : 20 &: \\ : 20 &: 8\,, \end{aligned}$$

or

$$\begin{aligned} \overline{x} : \overline{y} &: \overline{z} \\ 15 : 20 &: 8\,. \end{aligned} \tag{40}$$

Therefore, T must be a multiple of $\overline{x} + \overline{y} + \overline{z} = 43$, which again forces us to choose 86. But we see from (39) that by choosing $x = 15$, it's possible that $T = 43$, in which case the $\mathrm{GCD}(x, y, z) = 1$, although 43 is not in the list to choose from.

But before we jump to the conclusion that this alternative method is faster, it used some results from the prior method.

But what's up with these overbars? Well, in taking the common ratios of x, y, and z, we could have divided out the $\mathrm{GCD}(x, y, z)$. In fact, that's what happened, because their common factor is 2 or more, if we are to believe the choices given to us. Thus,

$$x + y + z = \mathrm{GCD}(x, y, z) \times (\overline{x} + \overline{y} + \overline{z})\,. \tag{41}$$

So, someone solving this problem on a timed exam, such as the GMAT, being quite expert at it, might by mere muscle memory arrive at the answer quickly by using this second method, but it has its own built-in subtleties. I think that a drawback one faces by emphasizing speed when solving word problems, is that they can forget the rationale behind the steps they take.

24 Word Problem #33.9

Problem (source unknown): After traveling a certain distance, a cyclist stopped for thirty minutes to reapir his bicycle. Then he continued the rest of his 30 km journey at half his original speed, with a total journey time of 5 hours. How far did he travel before he stopped to repair his bike?

25 33.8.1: Conceptualizing the Problem

Obviously, we can divide the cyclist's journey into three logical parts: First, the part where he travels at speed v_0, say. The second part he is not traveling and merely fixing his bike. The third, and last part, he travels the remaining distance, at a speed $v_0/2$.

I call these division 'logical' because they help us logically assess what's going on. Yes, they do even more: we can add up the time for each component to get a total time. And we can add up the component distances to get a total distance. Although, we can't add up the sum of the speeds to get a useful value.

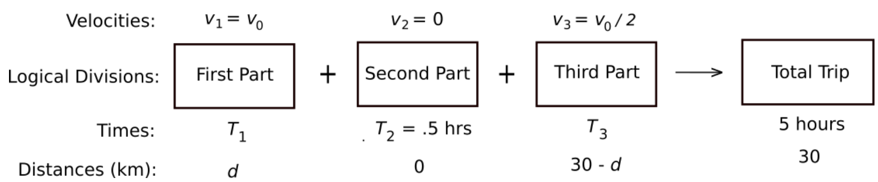

Figure 8. All the given data is already represented in the figure, but is it enough?

26 33.8.2: Solving the Problem

As I work this problem, I get four variables with only three equations to relate them. Maybe I did not get all the information I needed to write down this problem. In any case, I will add to the given information that initial speed of the cyclist is 8 km/hr. Hence,

$$v_0 = 8\text{km/hr}\,. \tag{42}$$

The sum of all the times is 5 hours:

$$\begin{aligned} T_1 + 0.5 + T_3 &= 5\,, \\ T_1 + T_3 &= 4.5\,. \end{aligned} \tag{43}$$

We've already accounted for the total distance traveled, but we have '$d = vt$' type relationships to write down. From the first part we have:

$$d = v_0 T_1 = 8T_1 \quad \longrightarrow \quad T_1 = d/8\,. \tag{44}$$

From the third part we have:

$$30 - d = \tfrac{1}{2} v_0 T_3 = 4T_3 \quad \longrightarrow \quad T_3 = (30 - d)/4\,. \tag{45}$$

Substituting these last two results into (43), we get

$$d/8 + (30 - d)/4 = 4.5\,. \tag{46}$$

On solving for d, the distance traveled prior to stopping to fix the bike,

$$d = 24\text{km}\,. \tag{47}$$

27 Conclusion

So, I get problems to solve for the reader's benefit from three main sources: First, from textbooks, second, from websites that deal in algebra problem solving (such as www.algebra.com and GMAT sites), and, third, from problems I make up myself. You may have observed that my solutions tend to be rather long, containing one or more diagrams and sometimes tables, and, in any case, a lot of exposition.

There are a few things throughout this series of articles that I wanted to convey to the reader. First, there is a systematic way to approach these kinds of algebra word problems, which I refer to as *Scheme*. Second, to really understand what you are doing in solving these tricky word problems, you need to flesh them out with diagrams and explicitly stated assumptions and logic, and, often, think of the processes involved as occurring in successive stages of development. Lastly, I wanted to confront and dispel the attitude that college algebra word problems are always trivial as certainly false. Some of them are quite tricky, even without requiring trigonometry, calculus, or Laplace transforms to solve.

One often has to massage the given information into a form that can be used in an algebraic equation, as we saw in the problems in which the given information is presented as ratios of one of more things to other things, such as can occur in the real-world problems of increasing or decreasing the oil-to-gas ratio of a mixture. For example, one's leaf blower may need a different oil-to-gas ratio than one's electrical gnerator, which itself may need a different ratio to a lawnmower.

Some of the problems I solve come from websites that receive either random algebra problems or problems of a type suitable for the GRE, the SAT, the GMAT, etc. Although most of the presented solutions on those websites are correct, they often do not come with much explanation — often little more than a collection of equations to solve.

Lastly, there will be times when one needs to make assumptions in order to solve a given problem. In other problems, one may be able to make simplifying approximations, without reducing the accuracy of one's answer. An example of this is Problem 6, in which we assumed the a small amount of salt dissolved in water will not appreciably affect the volume of the water. If this assumption were not true, then we should have to get the relevant information from some table on the physical chemistry of water, but such is usually not required of an algebra word problem. Still, when doing problems that give information in the form of percentages, one must be careful to ascertain if the percentages are by volume or by weight (mass). This can be particularly tricky if your book or teacher expects you to interpret a percentage by volume, even if this interpretation makes no sense. As a general rule in algebra problems: mass is always conserved (at some level of refinement), but there is no general law of conservation of volumes in a reaction, though, when mixing fluids together it's often the case that volumes are approximately conserved. Generally speaking, one can add a dense solid to a fluid and expect the conservation of volumes to be preserved. But if that solid dissolves in the fluid, it may not be so.