Word Problems 5: The Mixed-Rate Problems #2

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Abstract

In this algebra word problem note, we use the Scheme to solve our second attempt at what I refer to as a 'mixed-rate problem'. The usual name for these problems is 'mixture' problems, but my term is meant to imply a larger space of problems than just those that fit literal mixing of two or more tangible things together. In this type of problem, two or more 'machines' work together at generally different rates to produce subtotals that add to a total.

1 Introduction

In our last note on mixed-rates, we solved two word problems that both used literal machines in their formulations: one kind of machine being a printer, the other kind being an oil pump. In this note, the notion of machine is generalized: people working together at different rates to complete a job together. In future notes, the notion of 'machine' will become even more abstract.

2 Word Problem #5.1

Steve can mow a lawn in three hours and Joe can mow the same lawn in two hours. How long will each of them take to mow the lawn if they both work on it together, except that Joe works 20 minutes before Steve starts to work?

3 Solution 5.1.1: Conceptualizing the Problem

The solution is not difficult if we restrain ourselves from rushing off to find the 'key variables'. One obvious total is that Steven and Joe will work together to complete a total of one job.

$$1 \text{ job} = (\text{part of job done by Joe}) + (\text{part of job done by Steve})\,. \qquad (1)$$

For our next refinement, we use the average rate R at which each mower does the job and then multiply by the time he takes on the job.

$$1\text{ [job]} = R_J T_J + R_S T_S\,, \tag{2}$$

where $R_J = \dfrac{1}{2}\dfrac{\text{job}}{\text{hour}}$ and $R_S = \dfrac{1}{3}\dfrac{\text{job}}{\text{hour}}$. Substituting these into (2) and suppressing the units, we get

$$1 = \frac{1}{2}\,T_J + \frac{1}{3}\,T_S\,. \tag{3}$$

Now, we were given a simple relationship between T_J and T_S, namely, $T_J = T_S + \frac{1}{3}$, where the $1/3$ results from converting the twenty minutes to hours. However, I want to prove that you could also arrive at this equation by searching for all the totals and finding this one:

$$\begin{aligned}\text{Joe's total time} &= \begin{bmatrix}\text{part of Joe's total time}\\ \text{spent working alone}\end{bmatrix} + \begin{bmatrix}\text{part of Joe's total time}\\ \text{spent working with Steve}\end{bmatrix}\\ T_J &= \frac{1}{3} + T_S \end{aligned} \tag{4}$$

So, for the solutions we get, $T_S = 1$ hr and $T_J = (4/3)$ hr. Thus, Steve works one hour, and Joe works an hour and 20 minutes.

4 Word Problem #5.2

Question 261460[1]: Working together, Sara and Heidi can milk the cows in 2 hours. Working alone, Heidi takes 3 hours longer than Sara (working alone). How long will it take Heidi to milk the cows alone?

5 Solution 5.2.1: Conceptualizing the Problem

We begin with the now standard first level, describing when Sara and Heidi work together:

$$1\text{ job} = (\text{part of job done by Heidi}) + (\text{part of job done by Sara})\,. \tag{5}$$

For our next refinement, we use the average rate R [job/hour] at which each woman completes the job of milking all the cows.

$$1\text{ [job]} = R_H T_H + R_S T_S\,, \tag{6}$$

Since we are given that when they work together, they take 2 hours, then we can write that $T_S = T_H = 2$, and we get (suppressing units)

$$1 = R_H T_H + R_S T_S = (R_H + R_S)2\,. \tag{7}$$

[1]Found on algebra.com mixture problems

So, how do we relate this equation to what we've been asked to find? Well, what have we been asked to find? We've been asked to find how long it will take Heidi to milk the cows by herself. The time for her to do that is just R_H^{-1}, which has units of hours/job. We require one more relationship to obtain a second equation: We need to convert to an equation the sentence: Working alone, Heidi takes 3 hours longer than Sara (working alone). Therefore, put as an equation, we get

$$R_H^{-1} = R_S^{-1} + 3\,, \tag{8}$$

or, put into words, the hours/job for Heidi = (the hours/job for Sara) + (3 hours/job).

6 Solution 5.2.2: Solving the Problem

Let's make a variable substitution to simplify the equations: $x = R_H^{-1}$ and $y = R_S^{-1}$. Then (7) and (8) become

$$2x^{-1} + 2y^{-1} = 1 \quad \text{and} \quad x = y + 3\,. \tag{9}$$

Eliminating y between these equations, yields (after quite some algebra) the quadratic equation

$$x^2 - 7x + 6 = 0\,, \tag{10}$$

with possible solutions $x = 6, 1$[hours/job], the latter value being inconsistent with the given information.[2] Thus Heidi requires 6 hours to do the job by herself, and Sara requires only 3 hours to do the job by herself. [3]

7 Conclusion

This last problem of Heidi and Sara is the kind of problem that should be mastered and reviewed often to keep oneself fresh on the distinction between job/time and time/job, or similar inverse relations.

As a final comment on calculating average rates from given information, such as calculating the rate at which a 'machine' will do as job, given how long it takes to complete the whole job. If you think about it, it is not reasonable to assume that milk maids milk at a constant rate, or that painters paint at a constant rate, or even that printer machines print at a constant rate. The problem is, though, we can't solve these problems if we don't make this simplifying assumption of a constant rate from the given information. The formal justification for making this assumption is the Zeroth Rule of Problem Solving.

[2] When a root satisfies a derived quadratic equation but not the original constraints, the root is said to be *extraneous*. That happened here because, although $x = 1$ satisfies the quadratic, the implication is that the value for Sara's time would have to be -2 hours, violating the implicit constraint that all time solutions must be positive.

[3] I also solved this on wolframalpha.com by entering $(1/x) + (1/y) = 1/2, x = y + 3$. It wouldn't solve it algebraically when I used negative exponents.