Word Problems 7: The Mixed-Rate Problems #4

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November 28, 2016

Abstract

In this algebra word problem note, we use the Scheme to solve our fourth attempt at what I refer to as a 'mixed-rate problem'. In this type of problem, two or more 'machines' work together at generally different rates to produce subtotals that add to a total. Quantitative information can be given in the problem in various forms: percentages, fractional amounts, or by ratios. We have to know how to deal with each of them.

1 Introduction

In the first of our problems, our 'machines' are how two different concentrations of water-paint mixtures are combined to produce a third water-paint mixture. In the second problem, we need to determine the minimum number of workers needed to accomplish making both large and small cakes, which are produced at different rates. Problems like this one show us how different the algebra looks when the variables are over discrete sets, rather then over continuous variables.

2 Word Problem #7.1

A particular paint needs to be thinned with water in ratio 2 parts paint to 1.5 parts water. The painter comes to you with the confession that he mixed the two improperly, making 6 liters total in equal parts of water and paint. He asks if this can be fixed. You study the problem algebraically for a few mintues and determine that it can be. How?

3 Solution 7.1.1: Conceptualizing the problem

The first question we ask is, What does the process of fixing the paint mixture look like? The solution means to add either more paint of more water until the result is in the correct ratio. And, of course, we need to know precisely how much of this additive we must add. We haven't been told that we have access to any more paint, but we will assume that we have as much paint as needed to fix the mixture should that be the fix.

So, do we add paint or water? Have we thinned it too much or not enough? The first thing we need to do is learn how to convert a ratio into a fraction. When we say that (exactly) two things A and B are mixed in ratio $a : b$, we are comparing part a to part b. Say we want to know the fractional amount of a in the total mixture $a + b$. Obviously, the answer is $\frac{a}{a+b}$. The actual improperly combined mixture has the fractional amount of paint: $\frac{1}{1+1} = \frac{1}{2}$, but it's supposed to have $\frac{2}{2+3/2} = \frac{4}{7}$. Since $\frac{1}{2} < \frac{4}{7}$, we need to add paint, but how much? That's the kind of question that put the 'al' in algebra! Anyway, we'll let the variable x stand for the amount (in liters) of paint we need to add to the mixture to fix it.

In the figure below, we show a 'before and after' process of mixing paint. This type of graphic is going to become very famiiar to the reader, if he or she stays with this series of notes. It's basically a three-layer (occasionally four-layer) approach to organizing the data for clarity. The middle layer names or describes the objects in the boxes. The bottom layer is for quantities, either countable or measureable, such as volumes or weights. The top layer indicates rates of conversions, either in percentages, fractional amounts, or in ratios.

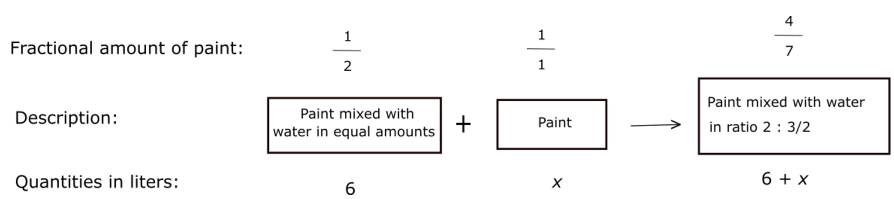

Figure 1. This graphic represents the mixing of paint in a 'before and after' process. The total paint before equals the total paint after. As noted in the figure, volumes are assumed to be conserved.

In a before-and-after process, we look for conserved quatities. By definition, a quantity is conserved if its total amount in the before state is equal to its total amount in its after state. The total of one or more things on the left will be equal to the respective total of that thing on the right. In this case, both total paint and total water are preseved. Also, the total of volume is asumed to be conserved in the process of mixing.[1] We've invoked the Zeroth Rule of Problem Solving to assume that the volumes add arithmetically, and the total of $6 + x$ has already been incorporated in the figure.

We begin with the conservation equation

$$(\text{total amount of paint before}) = (\text{total amount of paint after})\,. \qquad (1)$$

[1] We know for a fact that in some cases volumes of fluids do not add arithmetically, though the disrcepancies from naive expectation is usually so small they're negligible. We will assume that for our nonsceinctific use of this paint mixture, any diecrepancies are negligible.

For our next refinement, we expand the left hand side,

$$\begin{aligned}(\text{total amount of paint before}) = {} & (\text{part of paint in original mix}) \\ & + (\text{part of paint added in})\,. \end{aligned} \tag{2}$$

From this equation we get[2]

$$(\text{total amount of paint before}) = \frac{1}{2}(6) + 1(x) = 3 + x\,, \tag{3a}$$

where each term is a subtotal formed by taking a rate (from the top) times a quantity (from the bottom). For the amount of paint after in mixture 2:

$$(\text{total amount of paint after}) = \frac{4}{7}(6 + x)\,. \tag{3b}$$

On substituting the amounts from (3a) and (3b) into (1), we get the equation

$$3 + x = \frac{4}{7}(6 + x)\,, \tag{4}$$

which has solution $x = 1$ in liters.

So what's the general heuristic takeaway from this? If you identify a before-and-after process, look for all the conserved quantities and use as many of them as you need to solve for the unknowns you are looking for. We assumed there were three conserved quatities in this process: paint, volumes, and water, though we didn't use the fact that the water was conserved. But let's do so now.

We begin with the conservation equation for water

$$(\text{total amount of water before}) = (\text{total amount of water after})\,. \tag{5}$$

For our next refinement, we use

$$\begin{aligned}(\text{total amount of water before}) = {} & (\text{part of water in original mix}) \\ & + (\text{part of water in pure paint})\,. \end{aligned} \tag{6}$$

But there was no water in the pure paint added into the mix ($\frac{0}{1} = 0$), so for this equation we get

$$(\text{total amount of water before}) = \frac{1}{2}(6) + 0(x) = 3\,, \tag{7a}$$

Since the amount of paint in Mixture 2 is $\frac{4}{7}$, the amount of water in it is $\frac{3}{7}$ ($= 1 - \frac{4}{7}$), hence

$$(\text{total amount of water after}) = \frac{3}{7}(6 + x)\,. \tag{7b}$$

On substituting the amounts from (7a) and (7b) into (5), we get the equation

$$3 = \frac{3}{7}(6 + x)\,, \tag{8}$$

which also has solution $x = 1$ in liters.

[2]The fraction $\frac{1}{1}$ simplifies to just 1.

4 Word Problem #7.2

Workers are needed to help prepare 20 large cakes and 700 small calkes. Each worker can produce either 2 large cakes per hour or 35 small cakes per hour. What is the minimum number of workers needed to finish the job in 3 hours?

5 Solution 7.2.1: Conceptualizing the problem

We're going to begin with a few simplifying assumptions: We will employ three types of workers: The first kind will work exclusively on the large cakes (their number is L). The second kind will work exclusively on the small cakes (their number is S). And the third work on both kinds of cakes, if we need any such dual workers. Their number is B.

So,

$$(\text{total number of workers of all kinds}) = L + S + B\,, \tag{9}$$

where we stipulate that L, S, and B are nonnegative integers.

Here, we've encountered a rate with a difference from previous ones we've seen. In this problem the subtotals are proportional both to the number of workers and to the number of hours they work. This means that the

$$(\#\text{ of large cakes made by `}L\text{' people only}) = R_L \cdot L \cdot T\,, \tag{10a}$$

where T is the time in hours, and $R_L = \dfrac{2 \text{ large cakes}}{\text{worker}\cdot\text{hour}}$. Similarly,

$$(\#\text{ of small cakes made by `}S\text{' people only}) = R_S \cdot S \cdot T\,, \tag{10b}$$

where, again, T is the time in hours, and $R_S = \dfrac{35 \text{ small cakes}}{\text{worker}\cdot\text{hour}}$.

6 Solution Part 7.2.2: Solving the problem

We'll show the details of solving for L, and then S will follow similarly. The two solutions are actually independent of each other. Now, we solve (10a) for L:

$$L = \left\lfloor \frac{20 \text{ large cakes made by `}L\text{' people only}}{R_L \cdot T} \right\rfloor, \tag{11}$$

where we have optimistically put in all 20 large cakes in the hope that the result will be an integer and then we'll know what L is and that's it for the large cakes. However, if the fraction is not an integer, the so-called *floor function*[3] $\lfloor\cdot\rfloor$ will throw away the decimal part and what remains will be our L number. For example,

$$L = \lfloor 5.3241 \rfloor = 5\,. \tag{12}$$

[3]Technically, the floor function operates on a decimal number and returns the largest integer less than or equal it.

Plugging in our known values into (11), we get for L,

$$L = \left\lfloor \frac{\text{20 large cakes made by `L' people only}}{2 \cdot 3} \right\rfloor = \left\lfloor \frac{20}{6} \right\rfloor = 3\,. \tag{13}$$

So, how many large cakes can 3 ‘L’ workers make, each working 3 hours and each making 2 cakes per hour? Ans: 18. That leaves 2 large cakes to be done by the ‘B’ worker/s. Since we know that one worker can make 2 large cakes in one hour, we could get by with just 1 ‘B’ worker if the amount of time that worker will need for making small cakes is less than or equal to $3 - 1 = 2$ hours. Let’s find out.

For S, we get the equation

$$L = \left\lfloor \frac{\text{700 small cakes made by `S' people only}}{R_S \cdot T} \right\rfloor = \left\lfloor \frac{700}{35 \cdot 3} \right\rfloor = 6\,. \tag{14}$$

So, how many large cakes can 6 ‘S’ workers make, each working 3 hours and each making 35 cakes per hour? Ans: 630. This leaves 70 small cakes to be made by the ‘B’ worker/s. How long does it take for one worker to make 70 small cakes, making 35 in one hour and working for 3 hours? Ans: 2 hours.

Okay, we now know the result: We need 3 workers to work fulltime on large cakes; 6 workers to work fulltime on small cakes; and one worker, working for one hour to make 2 large cakes and for 2 hours to make 70 small cakes.

7 Conclusion

In this note I used a bit less of the English sentences in equation form, trusting that at this point in the series the reader can follow the presentation as is.