Fun with Fibonacci Numbers

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October 31, 2021

Abstract

In this survey paper we visit some fun theorems about the Fibonacci numbers, such as the Binet Formula and the Cassini Identity, and how to use generating functions to obtain the Binet Formula, and other surprises.

1 Introduction

My playful investigation of the Fibonacci numbers has led me into an amusement park of exhibits on them. One exhibit shows their connection to the Golden Ratio, another to a generating function for them, and another on the double uses of linear algebra to deal with them. Intrigued? If so, then step right up, ladies and gentlemen, and see the wonders of the humble Fibonacci numbers.

2 Getting Started

Leonardo Fibonacci was a thirteenth-century mathematician who helped to convince the Europeans to convert to the decimal number system we now enjoy in mathematics. But his fame comes from his publishing results on the so-called Fibonacci numbers, which had been know in antiquity to the mathematicians of India.

The Fibonacci sequence is based on the recursive definition: Starting with 0 and 1, add these two numbers to get 1.[1] Add the 1 and the 1 to get 2. Add the 1 and the 2 to get 3. Add the 2 and the 3 to get 5, and we have the start of an infinite sequence of numbers: $\{0, 1, 1, 2, 3, 5, \ldots\}$. The sequence has the simple recursive formula

$$F_{n+2} = F_n + F_{n+1} \,, \tag{1}$$

where $F_0 = 0$, $F_1 = 1$, $F_2 = 0 + 1 = 1$, $F_3 = 1 + 1 = 2$, $F_4 = 1 + 2 = 3$, and so on. Given that this sequence is defined over the nonnegative integers, it has no

[1] The literature on Fibonacci numbers shows the starting number as either 0 or 1, but the recurrence definition for these numbers is not affected.

ending and the numbers get big really fast:

$$0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, 377, 610,\\ 987, 1597, 2584, 4181, 6765, 10946, 17711,\\ 28657, 46368, 75025, 121393, 196418, 317811, \ldots \tag{2}$$

Let's first investigate the progressive consecutive ratios of these numbers, say, starting with 377:

$$610/377 = 1.618037, \quad 987/610 = 1.618033, \quad 1597/987 = 1.618034, \ldots \tag{3}$$

This new sequence of numbers looks like it may be approaching a famous number in mathematics, the **Golden Ratio**

$$\varphi = 1.61803398\ldots \quad \text{(not rounded)}. \tag{4}$$

Let's take a closer look at the Golden Ratio.[2]

According to the ancient Greeks, the Golden Ratio is said to occur as a relationship on the lengths of two adjacent sides of a rectangle[3] if, letting L stand for the length of the longer side and S the length of the shorter side, we get the proportion

$$\frac{L}{S} = \frac{L+S}{L}\,, \tag{5}$$

which is supposed to be a rectangle of maximal esthetic proportions (that is, of greatest beauty). Mind you, Equation (5) does not define the actual lengths of L and S, only their ratio.

Anyway, this equation has a numeric solution. Let $x = L/S$ and substitute back into (5) to get

$$x = 1 + x^{-1}\,. \tag{6}$$

Multiplying this through by x and rearranging terms, we get the usual form of a quadratic equation:

$$x^2 - x - 1 = 0\,. \tag{7}$$

The quadratic formula gives us the two irrational roots

$$\varphi_{\pm} = \frac{1 \pm \sqrt{5}}{2}\,, \tag{8a}$$

or in decimal form:

$$\varphi_{+} = 1.618033988\ldots \quad \varphi_{-} = -0.618033988\ldots\,. \tag{8b}$$

In Euclidean geometry φ_- is "unphysical," but who knows? In a spacetime metric, the Golden Ratio may only need an alternative interpretation of φ_+ as timelike and φ_- as spacelike, or something along those lines.

[2] According to Wikipedia, this relationship of the ratios of consecutive Fibonacci numbers to the Golden Ratio was first discovered by Johannes Kepler.

[3] We can talk about the Golden Ratio without mention of rectangles. Apparently the ancient Greeks divided line segments into the Golden Mean and used that idea to design features of the Parthenon, but this is a controversial claim.

3 Generating Function Approach

We naturally ask if we can come up with a nonrecursive form for the nth value of F_n.[4] One such form is the so-called Binet formula:

$$F_n = \frac{\varphi_+^n - \varphi_-^n}{\sqrt{5}}\,. \tag{9}$$

One way to discover a closed form is by the trick of employing a generating function.

A *generating function* is a formal power series whose coefficients are given by some sequence of numbers, particularly if there is a recurrence relation on the sequence. A generating function need not converge and we'll not take matters of convergence into account in this paper.

The idea of using a generating function is in two simple steps: First, can we take a combination of $f(x)$ with powers of x such that all but a finite number of terms survive? If we can, then, by use of algebra, we can recast $f(x)$ in a closed-form function of x.

Second, can we express $f(x)$ as a formal power series? If we can, then we equate the two formal series and then equate coefficients of respective powers of x, and, hopefully, obtain a formula for the nth coefficient. It turns out that we can do this with the Fibonacci sequence.

Let $f(x)$ be defined as follows

$$f(x) = \sum_{n=1}^{\infty} F_n x^n = F_1 x + F_2 x^2 + F_3 x^3 + \cdots\,, \tag{10}$$

where the coefficients are the Fibonacci numbers defined as above. We have ignored the zero term $F_0 = 0$, as it will not contribute to the sum. Now, the way to get the typical term of such a sum to cancel is to employ the recurrence relation (1) in the form

$$F_{n+2} - F_{n+1} - F_n = 0\,. \tag{11}$$

To get the two minus signs, we multiply $f(x)$ first by $-x$ and then second by $-x^2$:

$$\begin{aligned} f(x) &= F_1 x + F_2 x^2 + F_3 x^3 + F_4 x^4 \cdots \\ -x f(x) &= \qquad - F_1 x^2 - F_2 x^3 - F_3 x^4 - F_4 x^5 \cdots \\ -x^2 f(x) &= \qquad\qquad - F_1 x^3 - F_2 x^4 - F_3 x^5 - F_4 x^6 \cdots \end{aligned} \tag{12}$$

Now we just add them up and be amazed at all the cancellations:

$$(1 - x - x^2) f(x) = F_1 x + (F_2 - F_1) x^2 = x\,, \tag{13}$$

[4] This type of question is usual for mathematicians with or without a practical need to search for it. Yet, for practical considerations, wouldn't we rather have a closed form for the Fibonacci sequence to determine, say, $F_{1000000}$ rather than to compute it from the recurrence relation (1)?

where the quadratic term drops out because $F_1 = F_2$, and the higher-order terms drop out because of Equation (11). Solving for $f(x)$, we get

$$f(x) = \frac{x}{(1 - x - x^2)} = \frac{-x}{(x^2 + x - 1)}\,. \tag{14}$$

The following are a number of relations involving the φ_+ and φ_- we will make use of soon:

$$\varphi_+ + \varphi_- = 1\,, \qquad \varphi_+ - \varphi_- = \sqrt{5} \tag{15a}$$
$$\varphi_+^{-1} = -\varphi_-\,, \qquad \varphi_-^{-1} = -\varphi_+ \tag{15b}$$
$$\varphi_+\varphi_- = -1\,. \tag{15c}$$

The expression $x^2 + x - 1$ factors as $(x + \varphi_+)(x + \varphi_-)$. From here we use partial fractions to get $f(x)$ in the form of

$$\frac{-x}{x^2 + x - 1} = \frac{A}{(x + \varphi_+)} + \frac{B}{(x + \varphi_-)}\,. \tag{16}$$

My calculations give

$$A = -\frac{\varphi_+}{\sqrt{5}}\,, \qquad B = \frac{\varphi_-}{\sqrt{5}}\,. \tag{17}$$

Then

$$\begin{aligned} f(x) &= \frac{1}{\sqrt{5}}\Big[\frac{-\varphi_+}{(x + \varphi_+)} + \frac{\varphi_-}{(x + \varphi_-)}\Big] \\ &= \frac{1}{\sqrt{5}}\Big[\frac{-1}{(1 + y_+)} + \frac{1}{(1 + y_-)}\Big]\,, \end{aligned} \tag{18}$$

where $y_+ = x/\varphi_+$ and $y_- = x/\varphi_-$. Now, expanding these two terms in power series, we get

$$f(x) = \frac{1}{\sqrt{5}}\Big[\sum_{n=0}^{\infty} -(-1)^n y_+^n(x) + \sum_{n=0}^{\infty}(-1)^n y_-^n(x)\Big]\,. \tag{19}$$

Combining these summations and dropping out the zeroth terms, we get

$$f(x) = \frac{1}{\sqrt{5}}\sum_{n=1}^{\infty}(-1)^n[\varphi_-^{-n} - \varphi_+^{-n}]x^n\,. \tag{20}$$

Finally, using the results in (15b), we get

$$f(x) = \sum_{n=1}^{\infty}\Big[\frac{\varphi_+^n - \varphi_-^n}{\sqrt{5}}\Big]x^n\,. \tag{21}$$

Making a term-by-term comparison of this last series with (10), we can conclude that

$$F_n = \frac{\varphi_+^n - \varphi_-^n}{\sqrt{5}}\,. \tag{22}$$

As a practical matter, φ_-^n goes to zero quickly as n goes large.

4 Linear Algebra Approach

In this section we will reprove the Binet Formula and after that, prove the Cassini Identity using linear algebra.[5] It's assumed that the reader knows how to find eigenvalues/eigenvectors for a square matrix.

$\vdash$ **Binet Formula**

Beginning with a clever construction of the following matrices ($n \geq 2$)

$$\begin{bmatrix} F_1 \\ F_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} F_n \\ F_{n+1} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} F_{n-1} \\ F_n \end{bmatrix} . \tag{23}$$

We can stepwise reduce the right-hand equation to get

$$\begin{bmatrix} F_n \\ F_{n+1} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}^2 \begin{bmatrix} F_{n-2} \\ F_{n-1} \end{bmatrix} . \tag{24}$$

Repeating this process $n-1$ times gives[6]

$$\begin{bmatrix} F_n \\ F_{n+1} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}^{n-1} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \qquad (n \geq 1) . \tag{25}$$

As it stands, the $n-1$ matrix multiplications of (25) have not advanced the efficiency of this search for a shortcut to an arbitrary F_n, but what if we can diagonalize the square matrix? Let

$$M = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} . \tag{26}$$

Suppose M can be diagonalized in the usual way, by finding a *pachatti* matrix[7] P such that

$$D = P^{-1}MP = \begin{bmatrix} \alpha & 0 \\ 0 & \beta \end{bmatrix} , \tag{27}$$

where α and β are the eigenvalues of M, and P is constructed from the eigenvectors of M. Solving for M, we get

$$M = PDP^{-1} , \tag{28}$$

and thus

$$M^{n-1} = PD^{n-1}P^{-1} = P \begin{bmatrix} \alpha^{n-1} & 0 \\ 0 & \beta^{n-1} \end{bmatrix} P^{-1} . \tag{29}$$

[5] My resources for this were a number of web sites and a few YouTube videos, including one from Gilbert Strang of MIT.

[6] For an induction proof of this, see the Appendix.

[7] I know of no generally accepted name for this common matrix. So, until I find a better name, I'll use this invention of mine.

The characteristic equation of M is

$$\lambda^2 - \lambda - 1 = 0\,. \qquad (30)$$

But we have seen this equation before. It's the same as Equation (7) and thus has the same roots, which just happen to be the eigenvalues of M. Thus

$$D^{n-1} = \begin{bmatrix} \varphi_+^{n-1} & 0 \\ 0 & \varphi_-^{n-1} \end{bmatrix}\,, \qquad (31)$$

where, again, $\varphi_-^{n-1} \to 0$ as n goes large.

Acceptable eigenvectors[8] of M are $\begin{bmatrix} 1 \\ \varphi_+ \end{bmatrix}$ corresponding to $\alpha = \varphi_+$ eigenvalue, and $\begin{bmatrix} 1 \\ \varphi_- \end{bmatrix}$ corresponding to $\beta = \varphi_-$. Finally, the pachatti matrix is

$$P = \begin{bmatrix} 1 & 1 \\ \varphi_+ & \varphi_- \end{bmatrix}\,. \qquad (32)$$

And (25) becomes

$$\begin{bmatrix} F_n \\ F_{n+1} \end{bmatrix} = P \begin{bmatrix} \alpha^{n-1} & 0 \\ 0 & \beta^{n-1} \end{bmatrix} P^{-1} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \qquad (n \geq 1)\,, \qquad (33)$$

which has only three matrix multiplications.

⊢ Cassini Identity

The Cassini Identity is a relation on the Fibonacci numbers, given by

$$F_{n-1}F_{n+1} - F_n^2 = (-1)^n \qquad (n \geq 2)\,. \qquad (34)$$

The key step to this linear algebra proof of this identity begins with the observation that the left-hand side of (34) looks like the determinant of the matrix

$$B = \begin{bmatrix} F_{n-1} & F_n \\ F_n & F_{n+1} \end{bmatrix}\,. \qquad (35)$$

The question becomes: Can we find a convenient matrix equation involving B? The answer is yes, and for me it starts with (25).

LEMMA:

We will need the matrix inverse of M given in (26):

$$M^{-1} = \begin{bmatrix} -1 & 1 \\ 1 & 0 \end{bmatrix}\,. \qquad (36)$$

We note also the determinant of M:

$$\det(M) = -1\,. \qquad (37)$$

[8]Eigenvectors are determined up to a nonzero scale factor.

PROOF (Cassini Identity):

We will construct both columns of B out of (25):

$$B = \begin{bmatrix} F_{n-1} & F_n \\ F_n & F_{n+1} \end{bmatrix} = \left[M^{n-2} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \;\vdots\; M^{n-1} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right] . \tag{38}$$

Now we factor out M^{n-1}.

$$B = M^{n-1} \left[M^{-1} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \;\vdots\; \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right] = M^{n-1} \left[\begin{bmatrix} 0 \\ 1 \end{bmatrix} \;\vdots\; \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right] = M^{n-1} \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} = M^n . \tag{39}$$

This yields

$$\begin{bmatrix} F_{n-1} & F_n \\ F_n & F_{n+1} \end{bmatrix} = M^n . \tag{40}$$

Taking the determinant of both sides and using (37), we get

$$F_{n-1}F_{n+1} - F_n^2 = (-1)^n \qquad (n \geq 2) . \tag{41}$$

5 Conclusion

When I first saw the linear algebra proof of the Binet Formula for the Fibonacci numbers,[9] I was amazed. The takeaway heuristics from this paper is that if you can construct matrices (especially square matrices) out of your scalars, then you have a lot of ready-made structure from linear algebra waiting to be used from which to analyze your problem. This is evident both from the Binet and Cassini matrix proofs given here.

What a strange pathway through history we witness here, beginning with the ancient Greek notion of geometric beauty in their Golden Ratio to the Fibonacci numbers to proofs using eigenvalues and determinants of 2×2 matrices. I leave it to Naturalists to prove that this unlikely connection through history is all an accident.

6 Appendix: Induction Proof

The result we want to prove by induction is (25), restated below.

$$\begin{bmatrix} F_n \\ F_{n+1} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}^{n-1} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \qquad (n \geq 1) . \tag{42}$$

For $n = 1$, we get

$$\begin{bmatrix} F_1 \\ F_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} , \tag{43}$$

[9] I first saw the presentation of matrices to the Binet Formula (a few years ago) in Professor Strang's YouTube video series on linear algebra.

which is true. Next, we'll assume the result is true for $n = k$ and show that it's also true for $n = k + 1$. So, we assert the correctness of the following:

$$\begin{bmatrix} F_k \\ F_{k+1} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}^{k-1} \begin{bmatrix} 1 \\ 1 \end{bmatrix} . \tag{44}$$

Now, multiply through by M on the left:

$$\begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} F_k \\ F_{k+1} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}^{k} \begin{bmatrix} 1 \\ 1 \end{bmatrix} . \tag{45}$$

But

$$\begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} F_k \\ F_{k+1} \end{bmatrix} = \begin{bmatrix} F_{k+1} \\ F_k + F_{k+1} \end{bmatrix} = \begin{bmatrix} F_{k+1} \\ F_{k+2} \end{bmatrix} . \tag{46}$$

Putting these last two equations together, gives

$$\begin{bmatrix} F_{k+1} \\ F_{k+2} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}^{k} \begin{bmatrix} 1 \\ 1 \end{bmatrix} , \tag{47}$$

which is what we were to show.